- NO OH TODAY

- TAing for PACT (http://algorithmicthinking.org)
    - paid position for 4.5 weeks
    - email me, if interested

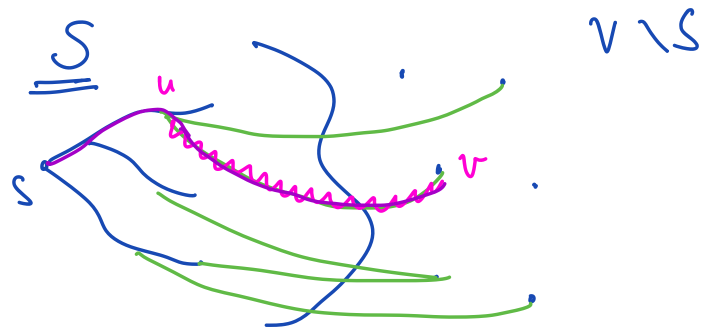
Shortest Paths

Input: Directed graph $G = (V, E)$

$s \in V$

wts on edges : non-negative

Obj : To find shortest paths from s to every other vertex in G.



Dijkstra (G, s) // assume that all vertices in G are reachable from s.

$O(n)$ for each $v \in V$ do

$d[v] \leftarrow \infty$ // distance estimate

$\pi[v] \leftarrow$ NIL

Use min-heap.

$\rightarrow$ key $\leftarrow d[\cdot]$ values.

$O(1)$ $d[s] \leftarrow 0$

$S \leftarrow \emptyset$

BuildHeap $O(n)$.

n times while $S \neq V$ do

n^2

$O(n)$ $\rightarrow$ $u \leftarrow$ vertex in $V \setminus S$ with the smallest $d[\cdot]$.

ExtractMin lg n.

$O(1)$ $S \leftarrow S \cup \{u\}$

for each $v \in \boxed{N(u) \cap (V \setminus S)}$ do

$\boxed{O(m)}$ over all iterations

if $d[v] > d[u] + w_{uv}$ then

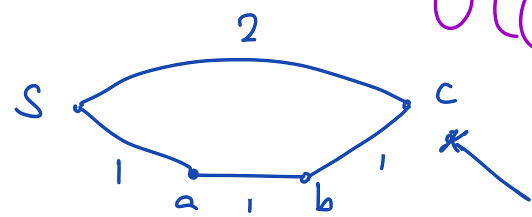
$d[v] \leftarrow d[u] + w_{uv}$ lg n.

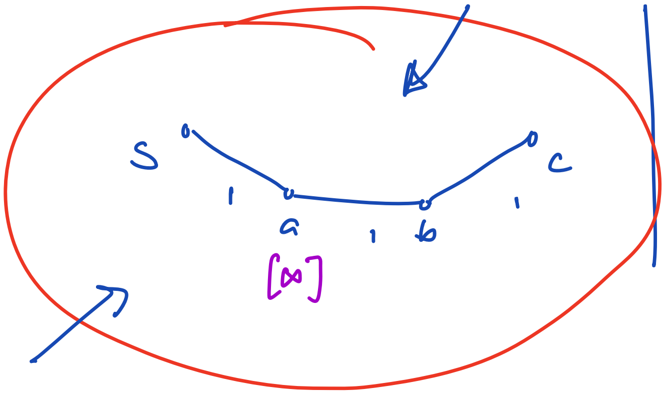
$\pi[v] \leftarrow u$ $\hookrightarrow$ Decreasekey

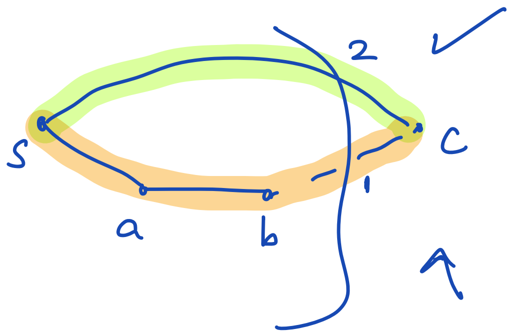
$O(n^2 + m)$

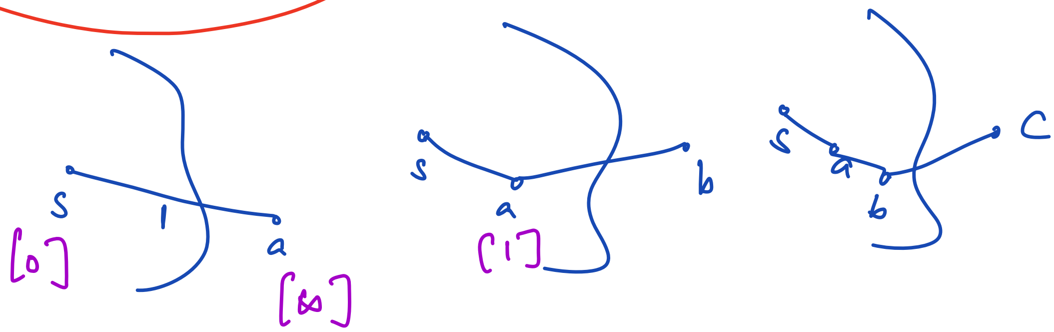
$O((n+m) \lg n)$

Example









Correctness

Theorem: Dijkstra correctly returns shortest paths for all vertices.

Proof: Induction on $|S|$.

IH: Assume that the claim holds when $|S| = k$. That is, when $|S| = k$, Dijkstra correctly computes shortest paths to all vertices in S.

BC : $|S| = 1$

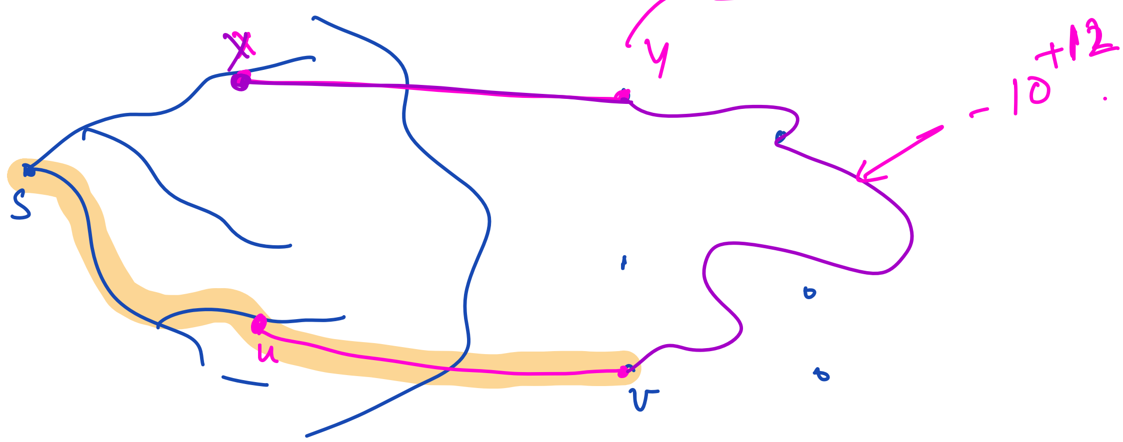
$S \ni s$. $d[s] = 0$ ✓

IS : We want to prove the claim when $|S| = k+1$. Let v be the $(k+1)^{th}$ vertex brought into S. Let $\pi[v] = u$. Thus $d[v] = d[u] + w_{uv}$.

Assume for contradiction that $d[v]$ is not the actual shortest path value from s to v. Instead the shortest path from s to v is

$$s \rightsquigarrow x \rightarrow y \rightsquigarrow v$$

$d[y]$ is very large



assume that edge wts are non-negative.

Thus

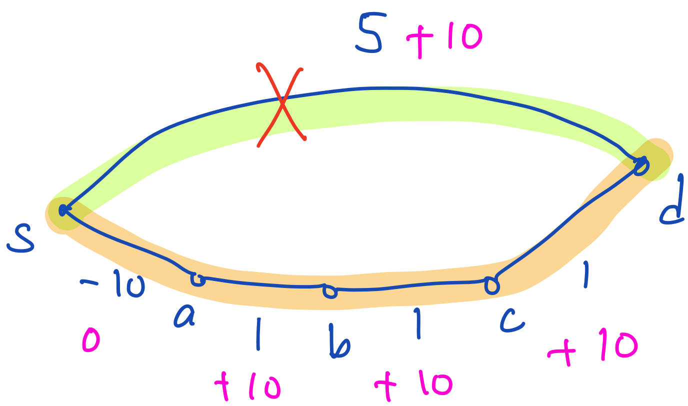
$$d[x] + w_{xy} + \ell(y,v) < d[v]$$

true only when edge wts are +ve

This means that $d[y] < d[v]$, which is a contradiction because, y would be brought in as the $(k+1)^{th}$ vertex & not v.

<u>fix</u>: Make all edge wts non-negative by adding the most neg. edge wt to all edge wts.

X



Minimum Spanning Trees (MST)

Input: Undirected graph $G = (V, E)$

wts on edges : positive.

Objective : To find a min wt spanning subgraph of G, that is connected.

↳ all vertices & no edges.

Lemma : The min wt. spanning subgraph of G that is connected must be a tree.

Assumption (WLOG): all edge wts are distinct.

<u>Algs</u> :

1. ~~Dijkstra~~

2. ~~BFS / DFS~~

3. Reverse Delete

   - Process edges in $\downarrow$ order of edge wts.
   - Delete an edge if removing it does not disconnect the graph.

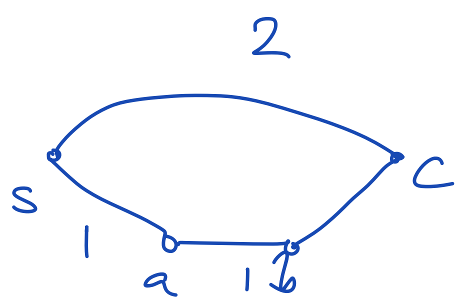
4. Kruskal

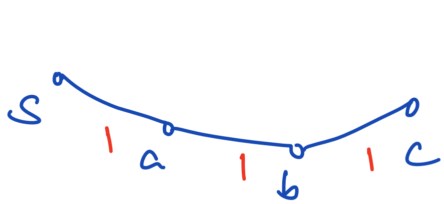
   - Process edges in $\nearrow$ order

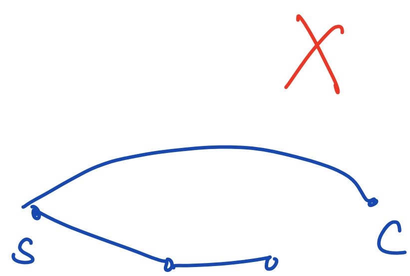
of wts

- Add an edge as long as adding it does not create a cycle.

⑤ Prim's alg. ("wrong Dijkstra")

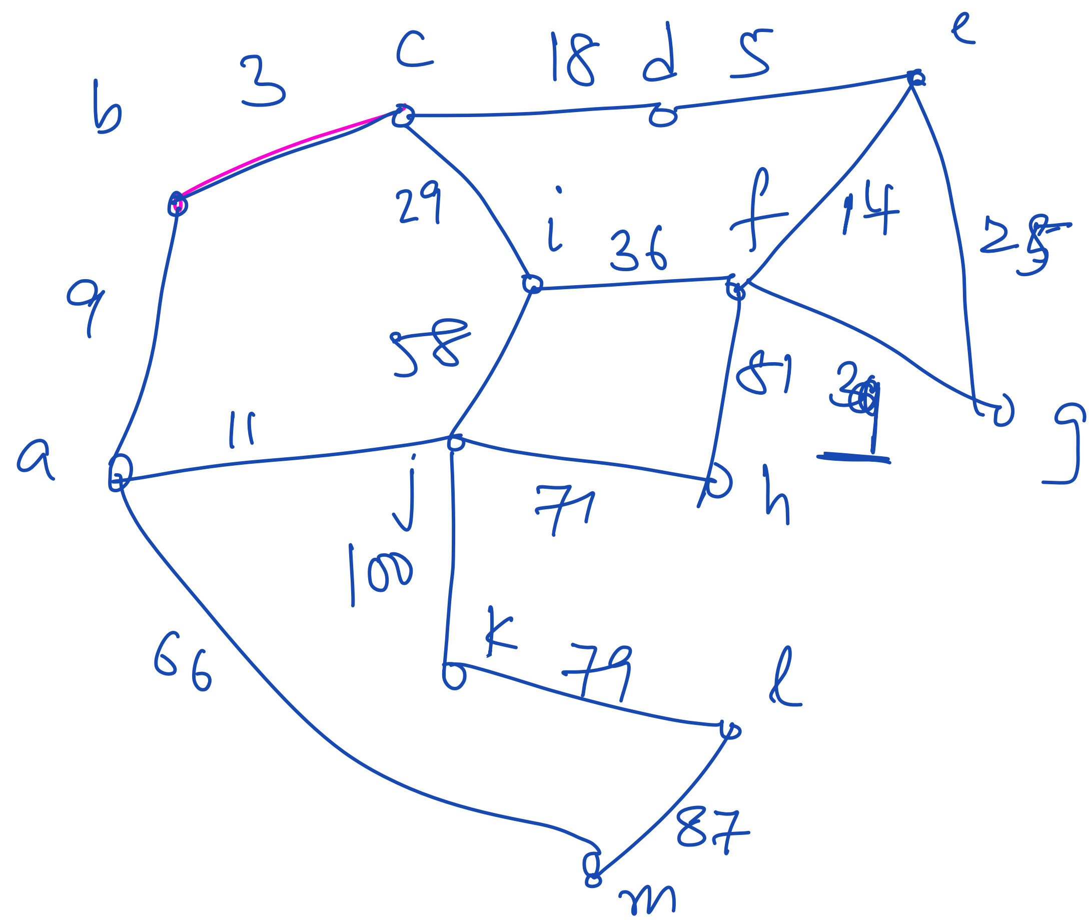


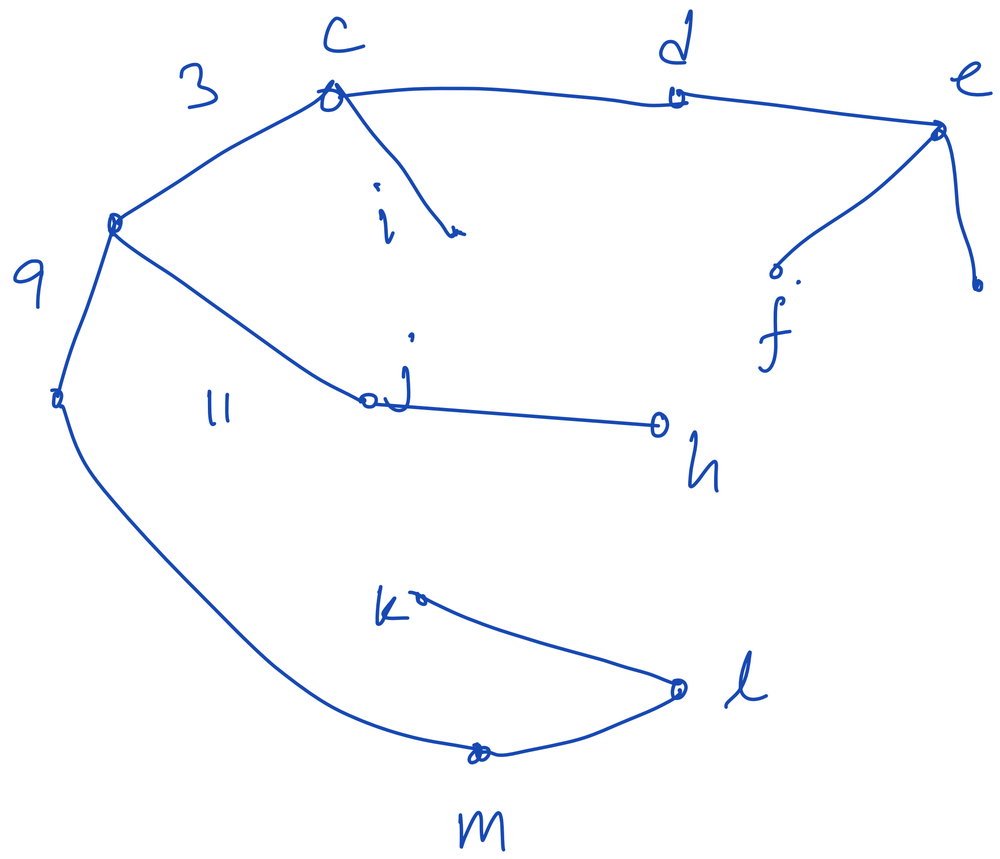




only change to Dijkstra

$$\left\{\begin{array}{l} \text{if } d[v] > w_{uv} \ \underline{\text{then}} \\ \quad d[v] \leftarrow w_{uv} \\ \quad \pi[v] \leftarrow u \end{array}\right.$$





Lemma: Let $(S, V \setminus S)$ be a partition of vertices in G, s.t. $S \neq \emptyset$ & $S \subset V$. Let $e = (u, v)$ be the min. wt.

edge crossing $(S, V \setminus S)$. Then e must be in every MST.

<u>Proof</u>: Assume for contradiction that there is a MST T that does not contain e. Let f be an edge in T that crosses $(S, V \setminus S)$. Note that f must exist. Consider

$$T' = T \setminus \{f\} \cup \{e\}$$

X Bogus!

Clearly, $wt(T') < wt(T)$

since $\boxed{w_e < w_f}$ & T' is

a spanning tree. A contradiction!