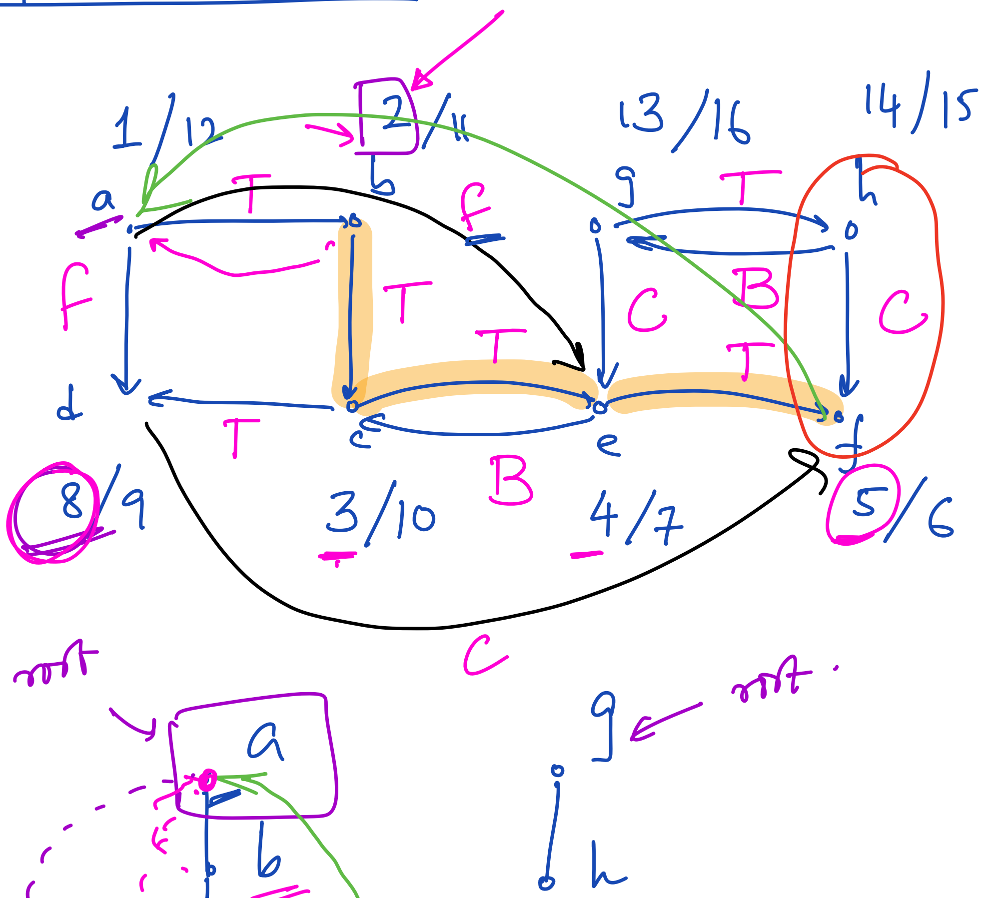
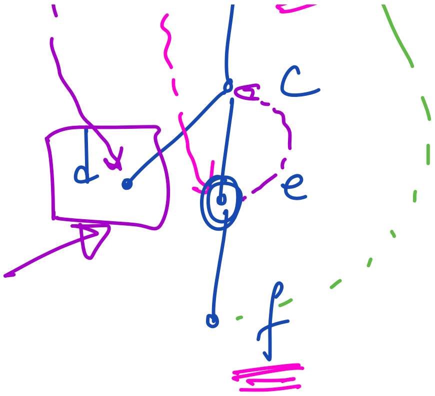
- No OH Today.

- Exam 1
  - Thu, Feb 27
  - seating on class page
  - recreate lectures / hws / recitations

Depth first Search





(u,v)

for each $v \in N(u)$

color[u] : white $\rightarrow$ gray $\rightarrow$ black

d[u] : discovery time of a vertex.

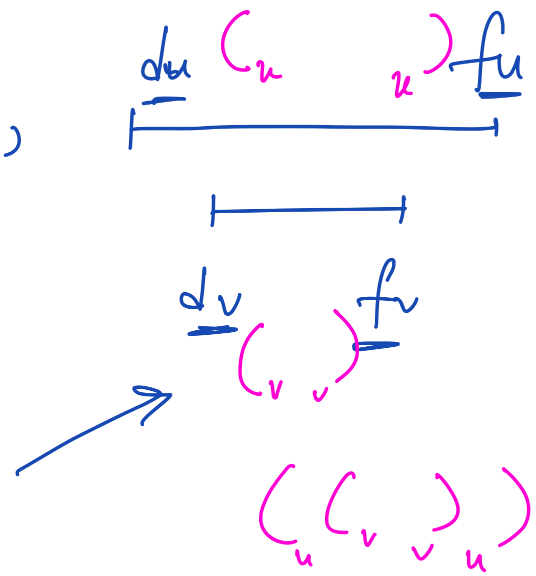
f[u] : finish time of a vertex.

Properties

When is a vertex v a descendant of vertex u in the DFS forest?

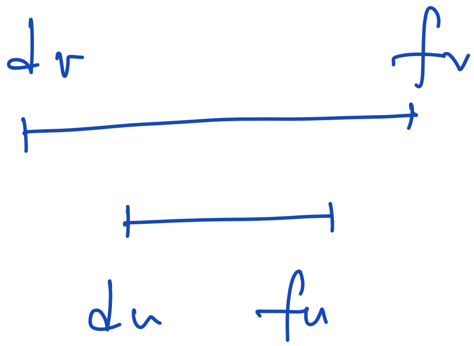
Property 1 : Vertex v is a descendant of vertex u in the DFS forest iff v is discovered when u is gray.

Property 2 (Parenthesis theorem) : Let u & v be vertices in G. Then exactly one of the following happens after a DFS(G).

(i) 

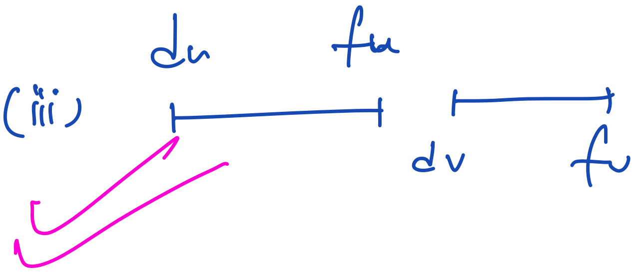
& vertex v is a descendant of vertex u in the DFS forest.

(ii)

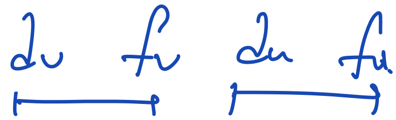


& vertex u is a descendant of vertex v in the DFS forest.

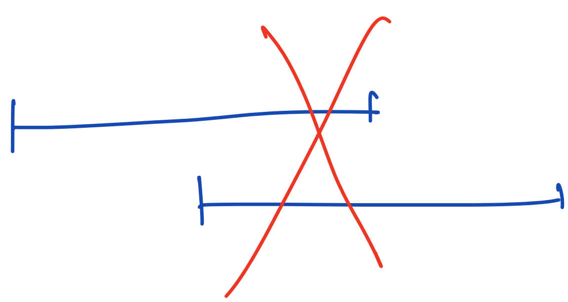
(iii)



or

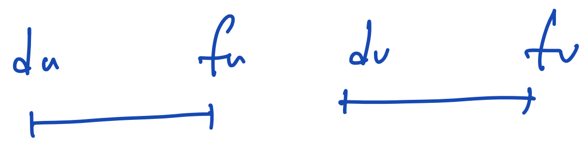


& neither vertex is a descendant of the other in the DFS forest.



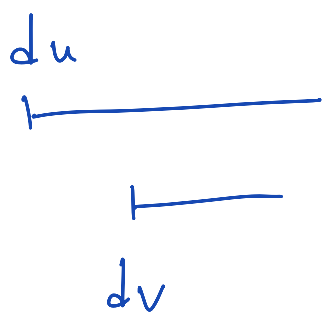
__Proof sketch__: WLOG, let $d[u] < d[v]$.

__Case I__: $d[v] > f[u]$



__Claim__: v is not a descendant of u in the DFS forest. (by property 1).

__Case II__: $d[v] < f[u]$



Since v is discovered when u is gray, v is a descendant of u in the DFS forest (by Property I).

It remains to show that $f_v < f_u$.

DFS "reaches" v, explores all its neighbors, colors v black before returning to u & hence $f[v] < f[u]$ ✓

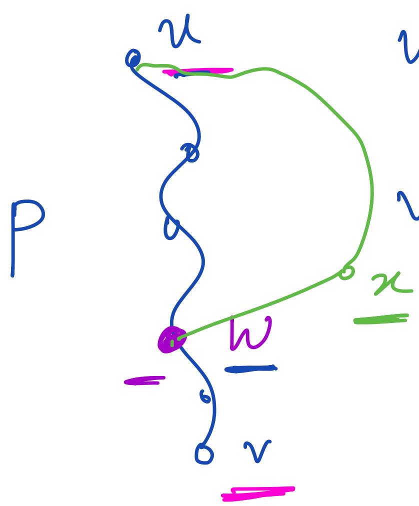
<u>Corollary</u>: Vertex v is a descendant of vertex u iff $d_u < d_v < f_v < f_u$.

<u>Property 3 (White Path Theorem)</u>. Vertex v is a descendant of vertex u in the DFS forest iff at time $d[u]$ there is a white path (path consisting

only of white vertices) from u to v in G.

Proof: ($\Rightarrow$) v is a descendant of u in the DFS forest $\Rightarrow$ at time $d(u)$, there is a white path from u to v in G.

Let P be the $u \rightsquigarrow v$ path in the DFS forest.



w: arbitrary vertex in P.

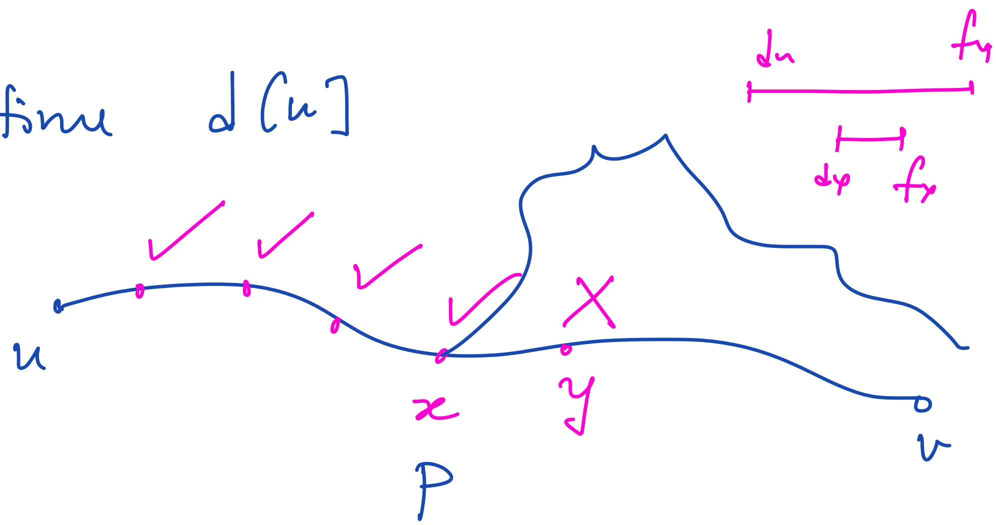
It suffices to show that at $d[u]$, w is white. By Property 1, w is discovered when color$[u]$ is gray. This means that when color$[u]$ was white, w is also white.

($\Leftarrow$) at time $d[u]$, there is a white path from u to v in G $\Rightarrow$ v is a descendant of u in the DFS forest.

Assume for contradiction that at time $d[u]$ there is a white path from u to

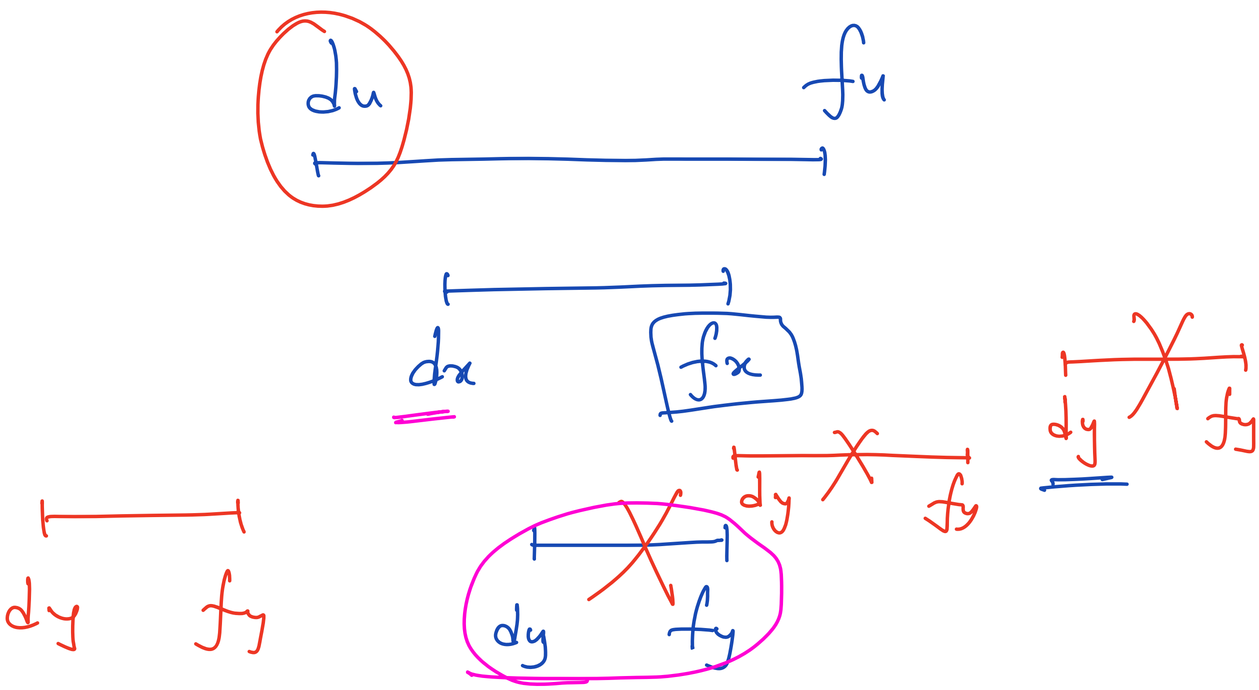
v in G but v is not a descendant of u in the DFS forest.

Let P be the $u \sim v$ white path at time $d[u]$



Going from u to v along P, let y be the <u>first vertex</u> that is not a descendant of u in the DFS forest. Let x be the vertex just before y in P. x is a descendant of u in the DFS forest.

By the Parenthesis thm



y is not a desc of u in the DFS forest.

By the Parenthesis thm, $[d_y, f_y]$ cannot be contained in the interval $[d_u, f_u]$. $d_y > f_u$ is also not possible because x could not have finished when it had a white neighbor in y. Thus $d_y < d_u$, This contradicts that at $d[u]$, y is white. ✓.

<u>Edge Classification</u> : We can classify edges in G w.r.t. a particular DFS on G.

That is, each edge receives exactly one label based on what happens when the edge is first explored. Consider an edge $e = (u, v)$ in G. e is a

tree edge : if e is in the DFS forest.

back edge : if v is an ancestor of u, when e is first explored.

forward edge : if v is a descendant of u when e is first explored.

cross edge : otherwise.

when e is first explored.

e = (u,v) is a tree edge. Then color[v] = white

back " " " gray

forward " black.

cross " black.

e = (u,v) is a forward edge in G iff

in DFS(G), color[v] is Black when

e is first explored & v is ~~a~~ descendant

of u in the DFS forest.

Theorem: DFS in an undirected

graph yields Tree edges

Back edges
forward edges
Cross edges.

<u>Proof sketch</u> : Let $e = (u, v)$ be an arb but particular edge in G.

WLOG, let $d_u < d_v$.

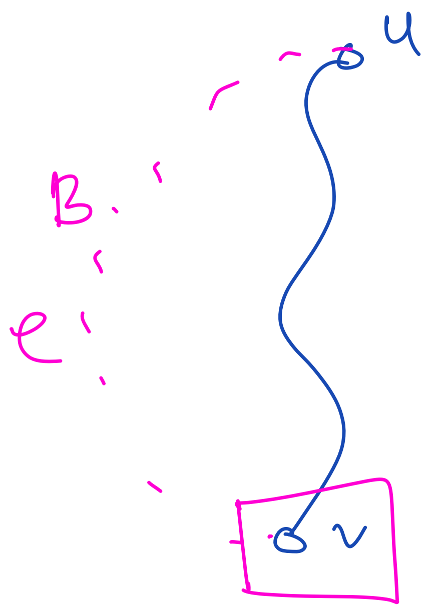
<u>Claim</u>: v is a descendant of u in the DFS forest.

<u>Proof</u>: White Path Theorem.

<u>Case I</u>: v is a child of u in the DFS forest.

Done, since e becomes a tree edge.

<u>Case II</u> : v is a descendant of u, but not a child in the DFS forest.



u is an ancestor of v when e is first explored.