CIS 160 - Fall 2012 **(instructor Val Tannen)**

Quizzes, with Answers and Explanations

QUIZ 1

1. Let X be a finite set with 100 elements and Y be a finite set with 2 elements. Then a function $f : X \to Y$ can be an injection, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. Because if f were an injection then $|X| \leq |Y|$, that is $100 \leq 2$ which is false. (By the way, the reasoning here is an example of proof by contradiction.)

2. Let A, B, C be three finite nonempty subsets of $\mathbb{R}$. Which of the following subsets of $\mathbb{R}$ may have a different size than the other two?

   (A) $(A - B) - C$

   (B) $(A - C) - B$

   (C) $A - (B \cap C)$

   Answer: (C). Because the sets in (A) and (B) are always equal (draw a potato diagram to convince yourselves of this), therefore they have the same size. This leaves the set in (C). To see that the set in (C) may have a different size then the (equal) ones in (A) and (B) note that those two also equal $A - (B \cup C)$ which can have a different size than $A - (B \cap C)$.

3. The number of sequences of length 3 whose elements are the lower-case letters of the English alphabet is

   (A) 26^3

   (B) 2^{26}

   (C) $26 \cdot 25 \cdot 24$

   (D) 3^{26}

   (E) $\binom{26}{3}$

   Answer: (A). Indeed, in each of the three positions of such a sequence we can put any of the 26 letters of the English alphabet. By the product rule we have $26 \times 26 \times 26 = 26^3$ such sequences. (A possible mistake is to choose (C) which is the number of *permutations* of 3 out of 26. But permutations are special sequences in which all elements are different.)

4. Let A, B, C be three finite nonempty subsets of $\mathbb{R}$. Then $(A \cup B) \cap C$ and $(A \cap B) \cup C$ can have the same size, true or false?

(TRUE)

(FALSE)

Answer: TRUE. There are various possible A, B, C's for which $(A \cup B) \cap C$ and $(A \cap B) \cup C$ have the same size. In fact, we can even make them equal. For example if we take $A = B = C$ or if we take $B = C = \emptyset$.

5. Let W be the set of all multiples of 3 between 1 and 100. The number of permutations of 2 elements from W is

(A) 1089

(B) 1122

(C) 1056

(D) 1156

Answer: (C). Indeed, the multiples of 3 between 1 and 100 are $3, 6, \ldots, 96, 99$ and there are 33 of them. The number of permutations of 2 out of 33 is $33 \times 32 = 1056$.

6. Let X, Y be nonempty subsets of $\mathbb{R}$ and consider the function $f : X \to Y$, $f(x) = x^2$. Then f can be a bijection, true or false?

(TRUE)

(FALSE)

Answer: TRUE. It can because, for example, X and Y could both be $[0, 1]$. For another example, X could be $[-2, -1]$ and Y could be $[1, 4]$.

(Notice that the question asks "can". So it suffices to show some example, as we just did. It does *not* ask "must". And indeed, there are plenty of X, Y for which f is not a bijection, for instance $X = \{-1, 1\}$ and take Y to be any set of real numbers containing 1. This discussion about "can" vs. "must" applies to several questions in this quiz.)

7. Let X be a nonempty subset of $\mathbb{N}$. Then a function $f : X \to X$ can be a surjection, true or false?

(TRUE)

(FALSE)

Answer: TRUE. It can because, for example, $f(x) = x$ defines a surjection.

8. Let $f : \mathbb{N} \to \mathbb{N}$ and let the subset $A \subseteq \mathbb{N}$ have 100 elements. Then the direct image $f(A)$ must also have 100 elements, true or false?

(TRUE)

(FALSE)

Answer: FALSE. It doesn't *have to* because, for example, if we define for any $x \in \mathbb{N}$, $f(x) = 13$ then $f(A) = \{13\}$ so it has just one element. (The "equal 13" function is a a "red herring", i.e., something that distracts from the real issue. The real issue is that the assignment of values to arguments that f does also defines another function, one with domain A and codomain $f(A)$ and that this other function is always a surjection. When $|A| = |f(A)|$ this function must in fact also be injective, that is, it must be a bijection. This tells us where to look for "counterexamples" like the one above.)

9. Let $f : \mathbb{N} \to \mathbb{N}$ and let A, B be finite subsets of $\mathbb{N}$ of 100 elements each. Then it is possible for $f(A \cup B)$ to have 2 elements while $f(A) \cup f(B)$ has 3 elements, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. It is not possible because we have shown in class that for any A, B we have $f(A \cup B) = f(A) \cup f(B)$ therefore these two sets cannot have different sizes.

10. Let X, Y be nonempty sets of 3 elements each. The number of surjections with domain X and codomain Y is

    (A) 3

    (B) 6

    (C) 9

    (D) 27

    Answer: (B). Let's say that $X = \{a_1, a_2, a_3\}$ and $Y = \{b_1, b_2, b_3\}$. A surjection with domain X and codomain Y must map each $a_i, i = 1, 2, 3$ to a *different* element of Y otherwise not all elements of Y will be mapped to. Then, the set of all the surjection from X to Y is in one-to-one correspondence with the permutations of $\{b_1, b_2, b_3\}$ ($a_i, i = 1, 2, 3$ is mapped by such a surjection to the element of Y that is in position i in the corresponding permutation). There are $3! = 6$ such permutations.

    (More generally, the argument above shows two statements worth remembering: (1) if the codomain of a surjection has as many elements as its domain than that surjection is actually a bijection, and (2) there are $n!$ bijections between two sets with n elements each.)

QUIZ 2

1. The number of sequences of bits such that four of the bits are 0, three of the bits are 1, and are such that they start with a 1 and end with two 0's is 16, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. Such sequences have a total of 4+3=7 bits but the first and the last two are fixed. This leaves 7-3 = 4 bits in the middle. If these 4 bits would be completely arbitrary then we would have $2^4 = 16$ possible sequences (and the answer would be TRUE). But they are not completely arbitrary: the sequence must have a total of four 0's. Out of these, two are at the end so exactly 4-2=2 must be among the 4 bits in the middle. The number of seuqnces of four bits with two 0's is $\binom{4}{2} = 6 \neq 16$ hence the answer is FALSE.

2. We count the rows and the positions in each row of the Pascal triangle starting from 1. In what row and what position is "50 choose 20"?

   (A) row 50 position 20

   (B) row 50 position 21

   (C) row 51 position 20

   (D) row 51 position 21

   Answer: (D). Because, as observed in recitation when we count rows and positions from 1 the number in row r and position p of the Pascal triangle is $\binom{r-1}{p-1}$.

3. Let X be a set with 2 elements and Y be a set with 3 elements. There are as many functional binary relations with domain X and codomain Y as there are functions with domain X and codomain Y, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. Because every function is a functional binary relation but not conversely; a binary relation is a function if it is both functional and *total*. But there exist functional relations with domain X and codomain Y which are functional but not total (for example the relation with the empty graph or the relation whose graph consists of just one pair). Therefore there are strictly more functional relations than functions.

4. Let X be a set with 2 elements and Y be a set with 3 elements. There are twice as many functions with domain $\text{pow}(X)$ and codomain $\text{pow}(Y)$ than binary relations with domain X and codomain Y, true or false?

(TRUE)

(FALSE)

Answer: FALSE. $|\text{pow}(X)| = 2^2$ and $|\text{pow}(Y)| = 2^3$ so there are $2^{3 \cdot 2^2} = 2^{12}$ functions with domain pow(X) and codomain pow(Y). On the other hand, there are as many relations as subsets of $X \times Y$ that is $2^{2 \cdot 3} = 2^6$. Clealry, 2^{12} is not $2 \cdot 2^6 = 2^7$.

5. The function $f : \mathbb{N} \to \mathbb{N}\ \ f(x) = 2^x - 1$ is a bijection, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. The function f is not a bijection because it is not a surjection: for example there is no x such that $f(x) = 2$.

QUIZ 3

1. Let $P(n)$ be a predicate where $n \in \mathbb{N}$. Suppose that we prove $P(0)$ and also $\forall k \geq 1\ P(k) \Rightarrow P(k+1)$. Then, by invoking the principle of ordinary induction we can conclude that $\forall n\ P(n)$, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. The restriction $k \geq 1$ in the induction step prevents us from concluding $\forall n\ P(n)$. It may be that $P(0)$ is true but $P(1)$, $P(2)$, etc. are all false (recall "false implies false" is true).

2. Let $P(n)$ be a predicate where $n \in \mathbb{N}$. Suppose that we prove $\forall n\ P(n)$ by ordinary induction on n and let S_o be the statement we prove in the induction step of such a proof. Suppose also that we prove $\forall n\ P(n)$ by strong induction on n and let S_s be the statement we prove in the induction step of such a proof. Then S_o implies S_s, true or false?

   (TRUE)

   (FALSE)

   Answer: TRUE. S_o is $\forall k\ P(k) \Rightarrow P(k+1)$ while S_s is $\forall k\ [\forall i,\ 0 \leq i \leq k,\ P(i)] \Rightarrow P(k+1)$. Now, if S_o holds then S_s holds because for any natural number k we have

   $$[\forall i,\ 0 \leq i \leq k,\ P(i)] \Rightarrow P(k) \Rightarrow P(k+1)$$

3. Assume that B is a set with 7 elements and that A is a finite nonempty set such that for any function $f : A \to B$ there exist at least 3 distinct elements of A that are mapped by f to the same element of B. Then the minimum number of elements that A can have is

(A) 3

(B) 8

(C) 15

(D) 22

Answer: (C). The generalized pigeonhole principle says that if $|A| > k \cdot |B|$ then $f : A \to B$ maps at least $k+1$ distinct elements of A to the same element of B. Here $k+1 = 3$ and $|B| = 7$ so we get $|A| > 2 \cdot 7 = 14$. It follows that A must have at least 15 elements.

4. Assume that A, B are finite nonempty sets and $f : A \to B$ is a function such that there exist at least 3 distinct elements of A that are mapped by f to the same element of B. Then $|A| > 2 \cdot |B|$, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. For example $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ and $f(a_1) = f(a_2) = f(a_3) = b_1$ but $|A| = 3 \leq 4 = 2 \cdot |B|$.

5. Let A, B, C be arbitrary finite sets and let $m = |A| + |B| + |C| - |A \cup B \cup C|$ and $n = |A \cap B| + |B \cap C| + |C \cap A|$.

   Then it is always the case that $m \leq n$, true or false?

   (TRUE)

   (FALSE)

   Answer: TRUE. Writing out the inclusion-exclusion formula for $|A \cup B \cup C|$ we can rearrange it as $n = m + |A \cap B \cap C|$. Since $|A \cap B \cap C| \geq 0$ it follows that $m \leq n$.

6. Consider the *uniform* finite probability space whose sample space consists of all pairs (my, dw) where my is a month of the year, i.e., January, February, ..., December and dw is a day of the week, i.e., Monday, Tuesday, ..., Sunday. The probability of the event "it's Wednesday" (that is, the set of all samples whose dw component is Wednesday) is

   (A) $\frac{1}{12}$

   (B) $\frac{1}{84}$

   (C) $\frac{19}{84}$

   (D) $\frac{1}{7}$

   Answer: (D). This probability space has $12 \cdot 7 = 84$ outcomes. Being uniform, each has probability $1/84$. The event "it's Wednesday" consists of 12 of these outcomes so it has probability $12/84 = 1/7$.

7. For any probability space and for any events E and F of that probability space we have $\Pr[E \cup F] \leq \Pr[E - F] + \Pr[F - E]$, true or false?

(TRUE)

(FALSE)

Answer: FALSE. Since $E \cup F = (E - F) \cup (F - E) \cup (E \cap F)$ and these are pairwise disjoint sets we have by the generalized Sum Rule

$$\Pr[E \cup F] = \Pr[E - F] + \Pr[F - E] + \Pr[E \cap F]$$

Therefore

$$\Pr[E \cup F] \leq \Pr[E - F] + \Pr[F - E]$$

is equivalent to $\Pr[E \cap F] \leq 0$. Clearly there are probability spaces and events for which this does not happen.

Note. One student astutely observed that an immediate and much easier proof is the counterexample obtained by taking $E = F = S$ (the entire sample space)!

8. Consider the *uniform* finite probability space whose sample space consists of all pairs (my, dw) where my is a month of the year, i.e., January, February, ..., December and dw is a day of the week, i.e., Monday, Tuesday, ..., Sunday. The probability of the event "it's April" (that is, the set of all samples whose my component is April) is

(A) $\frac{1}{12}$
(B) $\frac{1}{84}$
(C) $\frac{19}{84}$
(D) $\frac{1}{7}$

Answer: (A). This probability space has $12 \cdot 7 = 84$ outcomes. Being uniform, each has probability $1/84$. The event "it's April" consists of 7 of these outcomes so it has probability $7/84 = 1/12$.

9. For any probability space and for any events E, F, G of that probability space such that $\Pr[E \cap F] \neq 0$ we have $\Pr[E \cap F \cap G] = \Pr[E] \cdot \Pr[F|E] \cdot \Pr[G|E \cap F]$, true or false?

(TRUE)

(FALSE)

Answer: TRUE. By definition

$$\Pr[G|E \cap F] = \frac{\Pr[E \cap F \cap G]}{\Pr[E \cap F]} \qquad \Pr[F|E] = \frac{\Pr[E \cap F]}{\Pr[E]}$$

Multiply left hand and right hand sides of these two equalities.

$$\Pr[G|E \cap F] \cdot \Pr[F|E] = \frac{\Pr[E \cap F \cap G]}{\Pr[E \cap F]} \cdot \frac{\Pr[E \cap F]}{\Pr[E]}$$

and the desired equality follows. (Check that all denominators are non-zero.)

QUIZ 4

1. Consider a digraph with one vertex and no edges. Then, this digraph has a walk of length 0, true or false?

   (TRUE)

   (FALSE)

   Answer: TRUE. Let v be the vertex. The walk that starts at v, ends at v and has no edges in-between has length 0.

2. Consider a digraph $T_n = (V, E)$ where $|V| = n \geq 3$ and $E = V \times V$. The number of paths of length 2 in T_n is

   (A) $n(n-1)(n-2)$

   (B) n^4

   (C) $\binom{n}{3}$

   (D) n^3

   Answer: (A). We count the paths by the generalized product rule as follows. A path of length 2 has two edges and three *distinct* vertices. The start vertex s can be any vertex in V so it can be chosen in n ways. Once we have chosen s_1 note that there are edges from s_1 to all the vertices in V. However, we are counting paths, not walks, so we don't want the edge $(s_1 \to s_1)$. This means that the second vertex s_2 can be chosen in $n-1$ ways. From s_2 we can again go to any vertex but since we count paths we omit s_1 and s_2. Thus the third vertex can be chosen in $n-2$ ways hence the answer is (A). (By the way, if $n = 0, 1, 2$ the formula in (A) still gives the correct answer: there are 0 paths of length 2 in T_0 or T_1 or T_2. However, I put the condition $n \geq 3$ to remove the need to worry about these cases.)

   Remark By the way, looking at T_n, how many *walks* of length 2 does it have? To count thse, note that any edge can be the first edge of a walk of length 2 and there are n^2 edges. Once we have chosen the first edge, the second one has a fixed tail, namely the head of the first edge. However the head of the second edge can be any vertex so the the second edge can be chosen in n ways. The number of walks of length 2 is therefore $n^2 \times n = n^3$.

3. Let $G = (V, E)$ be a digraph such that $|V| = 3$ and $|E| = 2$. Then G must have at least one walk of length 2, true or false?

   (TRUE)

   (FALSE) *Answer:* FALSE. Consider $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_2), (v_3, v_2)\}$.

4. Consider a digraph $T_n = (V, E)$ where $|V| = n \geq 3$ and $E = V \times V$. Remove 1 (one) edge from T_n (do not remove any of the vertices). Then for any two distinct vertices s, t the length of the shortest path from s to t in the remaining graph is at most 2, true or false?

(TRUE)

(FALSE)

Answer: TRUE. Case 1: suppose we remove $(v, v'), v \neq v'$. For any other pair of distinct vertices there is an edge between them hence a path of length 1. For v, v', since there are at least 3 vertices, let v'' be another vertex (this is where we use $n \geq 3$, a condition that I forgot in the first version of the quiz!). Then the path $v(v \to v'')v''(v'' \to v')v'$ is a path of length 2 from v to v'. Overall, there are paths of length at most 2 between any two distinct vertices so the shortest paths also have length at most 2.

Case 2: suppose we remove (v, v). For any two distinct vertices the edge between them remains, hence there is still a path of length 1.

5. Consider the digraph $G = (V, E)$ where $V = 0, 1, ..., 9$ and where $E = \{(0,1),(1,2),(2,3),\ldots,(7,8),(8,9)\} \cup \{(9,8),(8,7),(7,6),\ldots,(2,1),(1,0)\}$. The number of paths of length 2 in G is

   (A) 9

   (B) 16

   (C) 8

   (D) 18

   Answer: (B). First count the paths that start with an edge of the form $(i, i+1)$. Then the second edge must be $(i+1, i+2)$ (cannot go back to i since we count paths, not walks) so we have such paths exactly for $i = 0, 1, \ldots, 7$ a total of 8.

   Next count the paths that start with an edge of the form $(i, i-1)$. Then the second edge must be $(i-1, i-2)$ (again cannot go back to i since we count paths, not walks) so we have such paths exactly for $i = 9, , \ldots, 3, 2$ also a total of 8.

   $8 + 8 = 16$

QUIZ 5

1. Consider a poset with exactly two minimal elements. Then, one of these two elements is minimum, true or false?

   (TRUE)

   (FALSE)

   Answer: FALSE. Let a, b such that $a \neq b$ be the two minimal elements. If a is minimum then $a \leq b$ so b is not minimal. Similar for b, contradiction in both cases.

2. Consider the set $X = \{a, b, c\}$ and the poset $(\text{pow}(X), \subseteq)$. The number of chains with four elements in this poset is

(A) 4

(B) 6

(C) 8

(D) 10

Answer: (B). Every chain of 4 elements has the form $\emptyset \subset S \subset T \subset \{a, b, c\}$. It follows that S is a one-element subset and T is a two-element subset. There are 9 pairs (S, T) but in 3 of them $S \not\subseteq T$. This leaves 6 good pairs therefore we have 6 chains of 4 elements.

3. Every total order on a finite set has both a minimum and a maximum element, true or false?

(TRUE)

(FALSE)

Answer: TRUE. Suppose that there is no minimum element. In a total order any minimal element is also minimum. So there cannot be any minimal elements either. Now let a be an element of the total order. Since a_0 is not minimal there exists $a_1 \neq a_0$ with $a_0 \geq a_1$. Now repeat the reasoning for a_1 and so on. We obtain that there must exist an subset of of *distinct* elements $a_0 \geq a_1 \geq a_2 \geq \cdots$. This contradicts the fact that the set is finite. Similar for maximum.

4. Let $G = (V, E)$ be a DAG where $V = \{a, b, c\}$. Recall that E^+, the transitive closure of E, is a strict partial order on V. Suppose that the *only* nonempty antichain in this strict partial order is $\{b, c\}$. Then a must be a minimum, true or false?

(TRUE)

(FALSE)

Answer: FALSE. Since $\{a, b\}$ and $\{a, c\}$ are not antichains a must be comparable with both b and c. Moreover it must be either smaller than both or bigger than both, otherwise b and c become comparable. So a must be either a minimum or a maximum.

5. There exist undirected graphs with 100 vertices, 50 connected components, and 49 edges, true or false?

(TRUE)

(FALSE)

Answer: FALSE. By the theorem in the textbook/class $49 \geq 100 - 50 = 50$ which does not hold so such a graph cannot exist.

6. Let $G = (V, E)$ be an undirected graph such that every node in V has degree 0 or 1. Denote by $C(G)$ the number of connected components of G. Only one of the following statements must be true for all such graphs. Which one?

(A) $|E| < |V| - C(G)$

(B) $C(G) = |E|$

(C) $C(G) = |V|$

(D) $|E| \leq |V|/2$

Answer: (D). (A) is false because it contradicts the theorem in textbook/class. (B) and (C) both fail for the graph $V = \{a, b, c\}$, $E = \{\{a, b\}\}$ because it has 3 nodes, 1 edge, and 2 connected components. (D) is true because the number of nodes of degree 1 is $2|E|$.

7. Let $n \geq 3$. Recall the graph C_n, its set of vertices is $\{1, \ldots, n\}$ and its set of edges is $\{\{1, 2\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. The number of different spanning trees of C_n is

(A) n

(B) $n - 1$

(C) n^2

Answer: (A). The spanning trees are obtained by removing an edge from C_n. There are n edges therefore n spanning trees.

8. For any undirected graph $G = (V, E)$ let's denote by $\text{diam}(G)$ the length of the longest path in G. Then $\text{diam}(G) \leq |V| - 1$, true or false?

(TRUE)

(FALSE)

Answer: TRUE. The vertices of a path must be all distinct so there can be at most $|V|$ of them. A path with $|V|$ vertices has length $|V| - 1$. So all the paths in G, including the longest ones, have length $\leq |V| - 1$.

9. For any undirected graph $G = (V, E)$ let's denote by $\text{diam}(G)$ the length of the longest path in G. Consider the set $\{ \text{diam}(T) \mid T \text{ is a tree with 100 nodes} \}$. Let N be the least number in this set (it exists by the well-ordering principle). N equals

(A) 1

(B) 2

(C) 99

Answer: (B). Because there is no tree with 100 nodes and diameter 1 (in fact, any tree with 3 or more nodes must have a path of length 2) but there exist a tree with 100 nodes and diameter 2: it's a "star" with a node m in the middle and each of the other 99 nodes connected to m by an edge.

10. Let $n \geq 3$. Recall the graph K_n, its set of vertices is $\{1, \ldots, n\}$ and its set of edges is $\{ \{i, j\} \mid 1 \leq i < j \leq n \}$. For any undirected graph $G = (V, E)$ let's denote by $\mathrm{diam}(G)$ the length of the longest path in G. Then $\mathrm{diam}(K_n) = n - 1$, true or false?

(TRUE)

(FALSE)

Answer: TRUE. We know from the question 8 above that $\mathrm{diam}(K_n) \leq n - 1$ so it suffices to show that there exists a path of length $n - 1$. In K_n any ordered list of the n vertices gives such a path.