Verification of Machine Learning Programs

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Summer School on Foundations of Programming and Software Systems
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Background

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  - Phones, airplanes, hospitals

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- Complexity is increasing
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  - Autonomous driving
- Manually creating software is *very* difficult

Machine Learning to the Rescue

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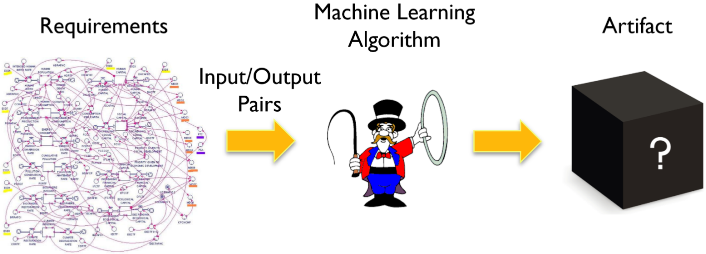

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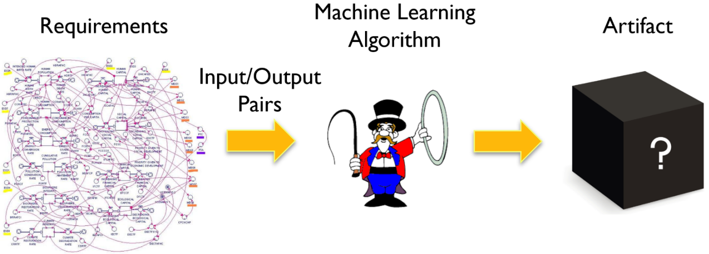

- Image recognition, game playing, autonomous driving, etc.

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- But their opaqueness can be dangerous
- Traditional quality-assurance techniques do not apply
  - Code reviews? Refactoring? Invariants?
- How do we know what is going on inside the black box?

When Things go Wrong...

The ACAS Xu System

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- Produce an advisory:
  - *Clear-of-conflict (COC)*
  - *Strong left*
  - *Weak left*
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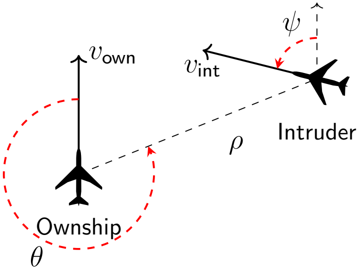

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  - Especially because this is a new approach

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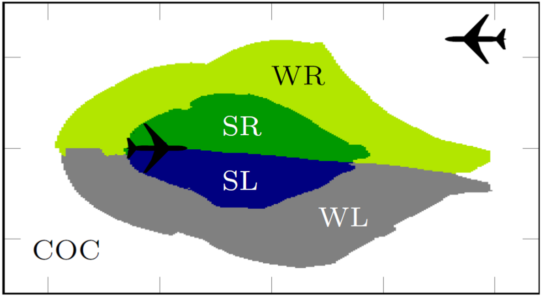

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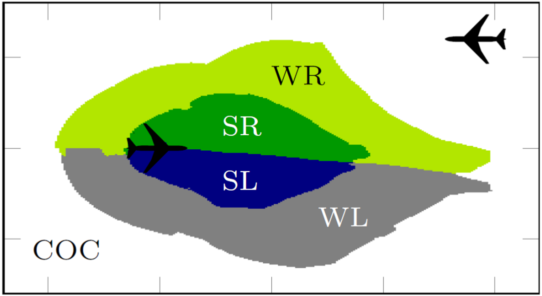

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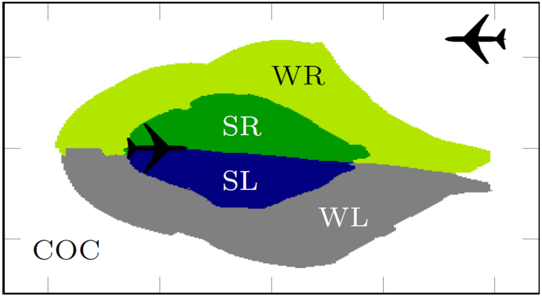

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  - Verification can help

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- But, computational cost much higher

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- Is it worth the effort?
  - *Yes*, especially for safety-critical systems (like ACAS Xu)

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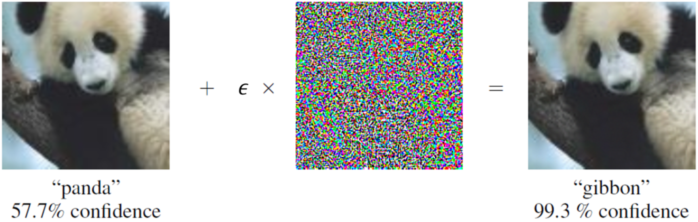

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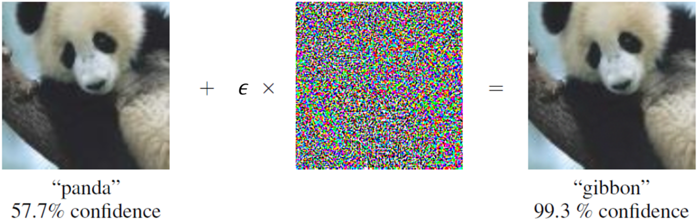

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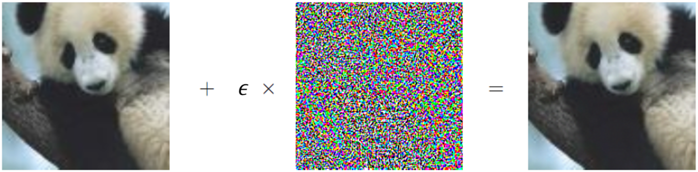

"panda"
57.7% confidence

"gibbon"
99.3 % confidence

- *Small perturbations* of inputs lead to misclassification
- Can usually find such inputs *very* easily

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  - Adversary changes "stop" sign into a "entering highway" sign?

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- Verification can be used to establish robustness *guarantees*

Roadmap

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  1. See why neural network verification is hard
  2. Survey state-of-the-art verification techniques
  3. Discuss one technique (Reluplex) in more detail

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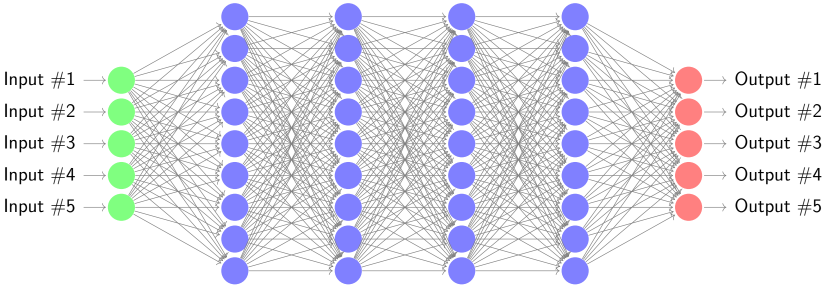
Neural Networks



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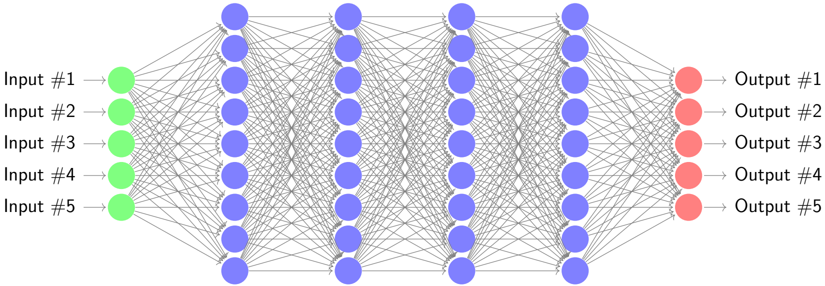

- Typical sizes (number of neurons): between few hundreds and millions

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- Each edge is assigned a *weight*, and these define the network's behavior

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- We assume that the network has already been trained

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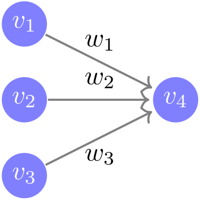

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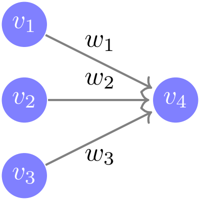

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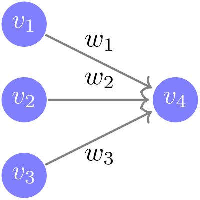

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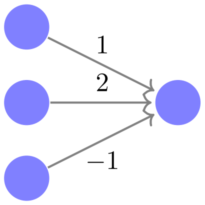

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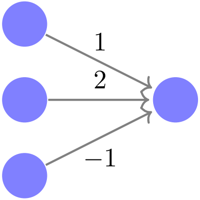

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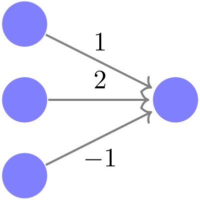

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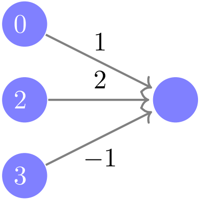

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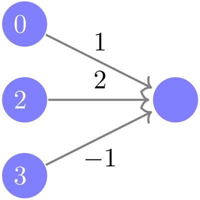

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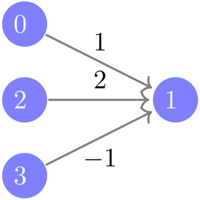

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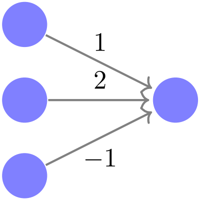

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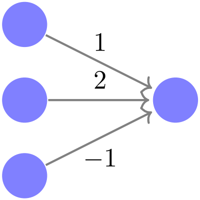

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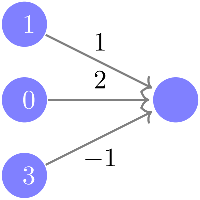

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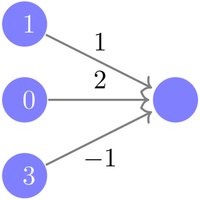

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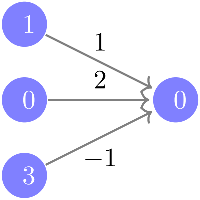

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- Hyperbolic tangent function: $f(x) = \mathsf{tanh}(x)$

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Definition (The Neural Network Verification Problem)

For a neural network $N : \bar{x} \rightarrow \bar{y}$, an input property $P(\bar{x})$ and an output property $Q(\bar{y})$, does there exist an input $\bar{x}_0$ with output $\bar{y}_0 = N(\bar{x}_0)$, such that $\bar{x}_0$ satisfies P and $\bar{y}_0$ satisfies Q?

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- We will construct an input to the verification problem that is satisfiable iff the formula is satisfiable

Reduction: Handling Negations

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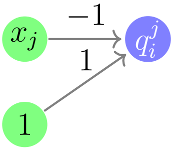

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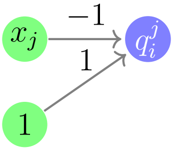

- q_i^j gets $1 - x_j$, i.e. $q_i^j = \neg x_j$

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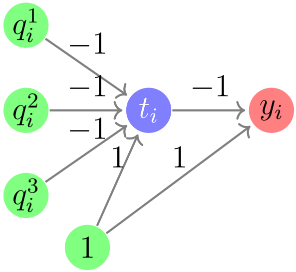

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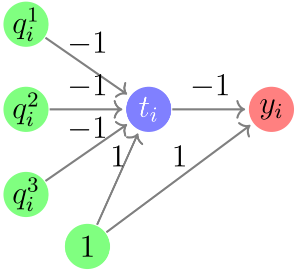

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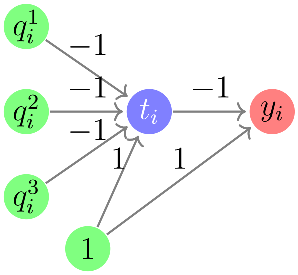

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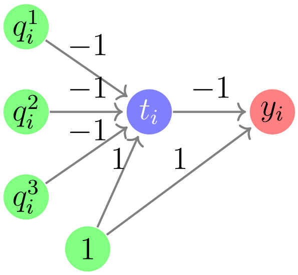

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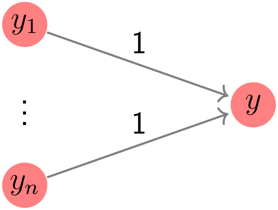

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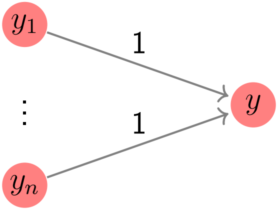

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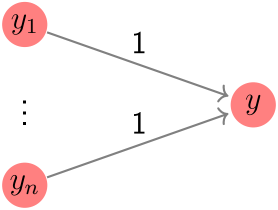

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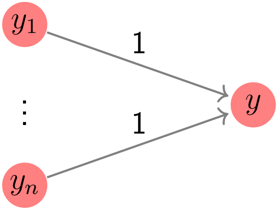

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- This is satisfied only if all conjuncts are 1

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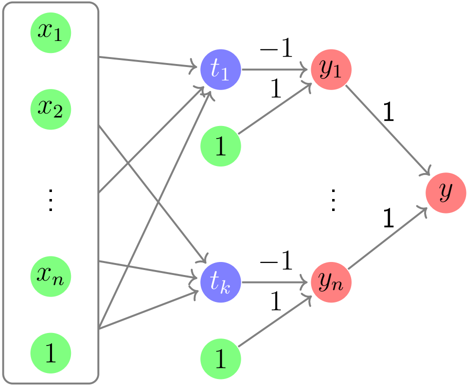

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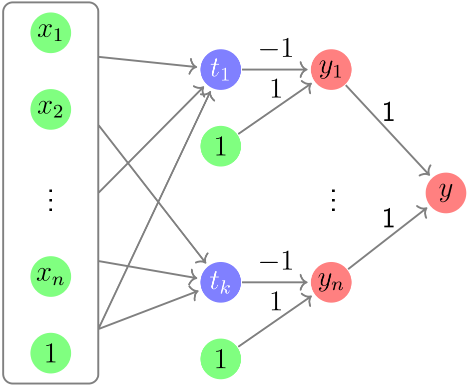

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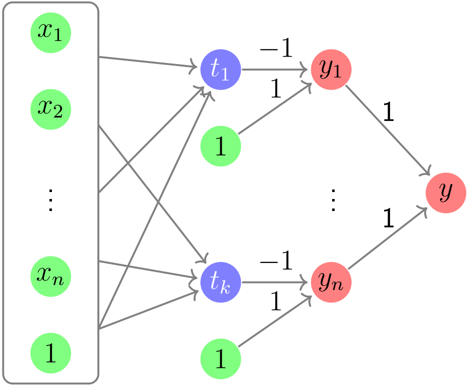

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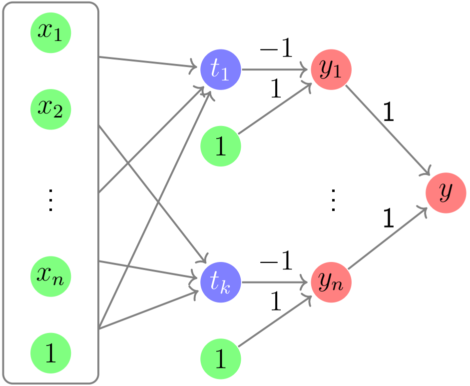

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Table of Contents

Disclaimer: The literature on neural network verification is growing rapidly. The work mentioned here is just a sample. Apologies to all authors whose work is not cited.

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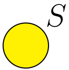

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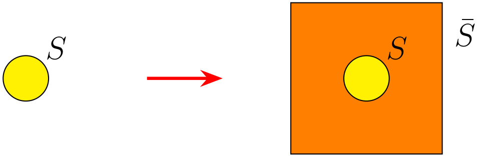

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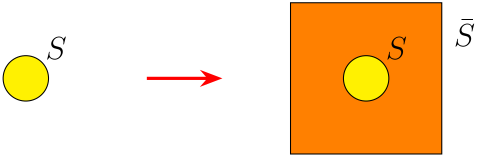

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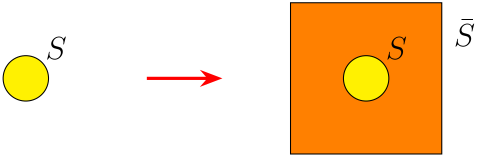

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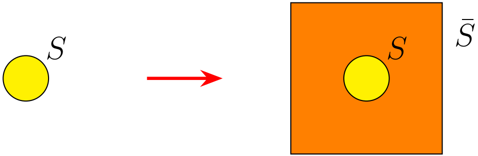

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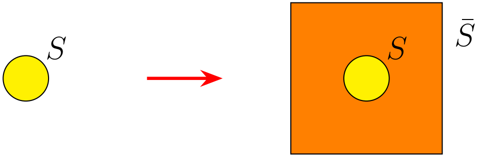

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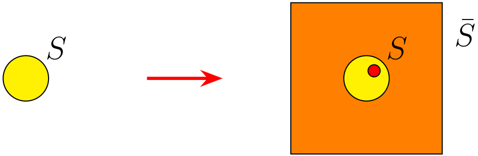

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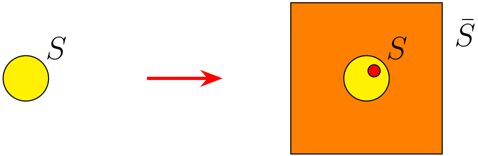

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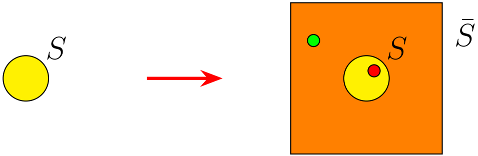

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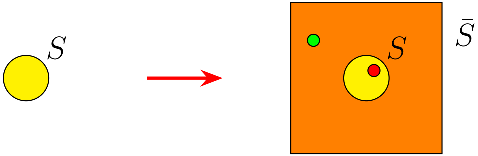

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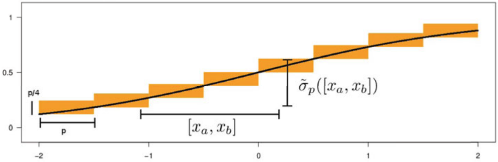

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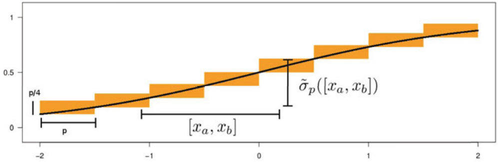

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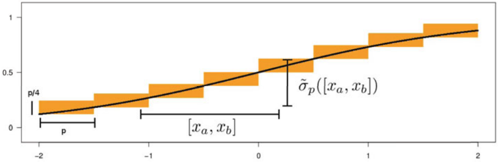

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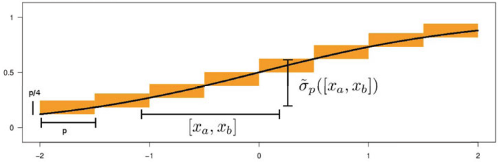

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- If a ReLU is known to be in a specific phase, it can be discarded and *replaced* with a linear equation

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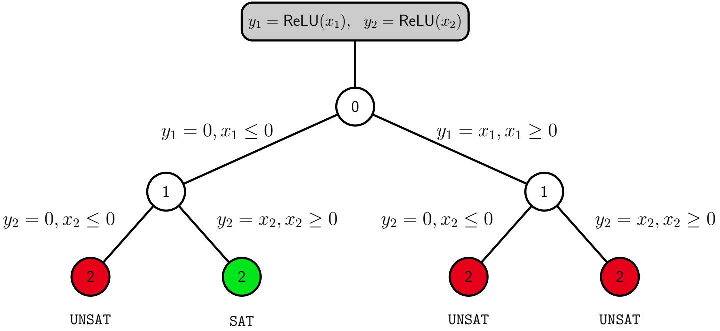

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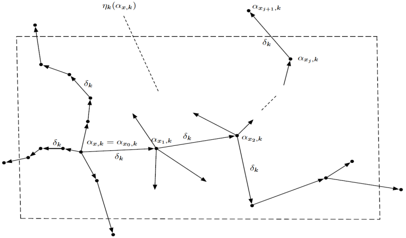

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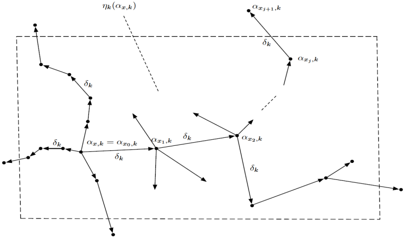

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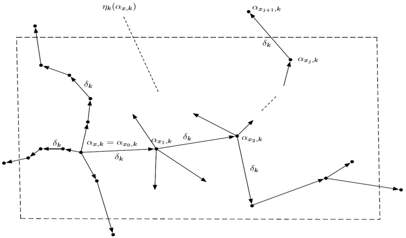

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- Tool: the *DLV* solver, evaluated on image recognition networks

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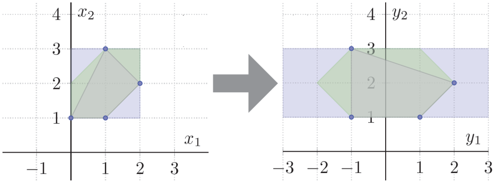

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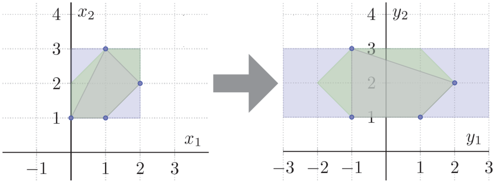

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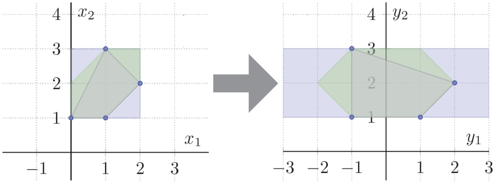

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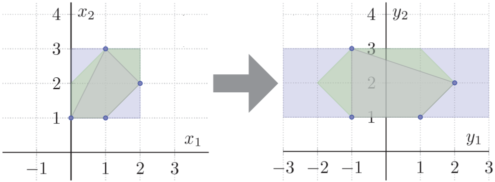

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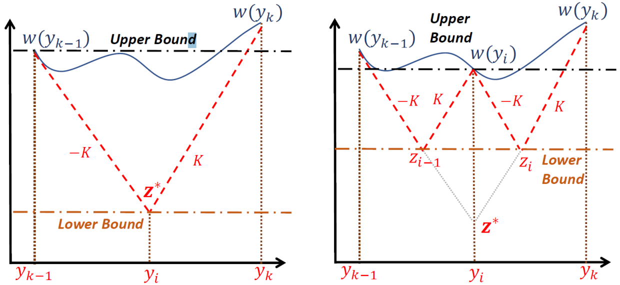

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Table of Contents

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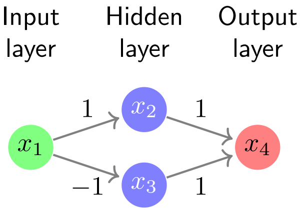

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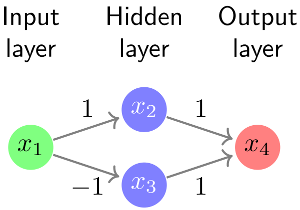

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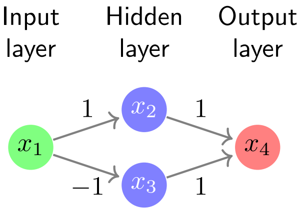

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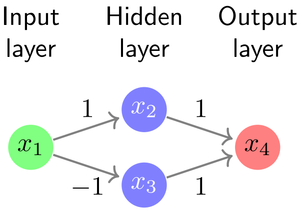

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  - Negated output property: $x_1 \in [0, 1]$ and $x_4 \in [0.5, 1]$

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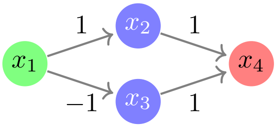

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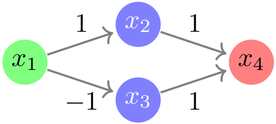

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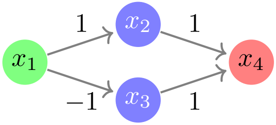

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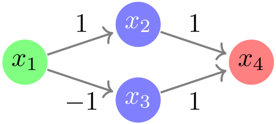

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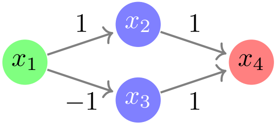

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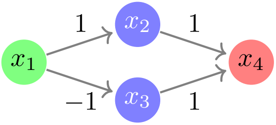

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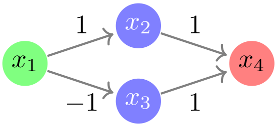

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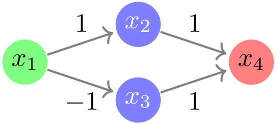

- Equations for weighted sums:

$$x_2 - x_1 = x_5$$
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	x_2	0	
	x_3	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
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Update:
$x_4 := x_4 + 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Simplex: Example (cnt'd)

$$x_5 = x_2 - x_1$$
$$x_6 = x_3 + x_1$$
$$x_7 = x_4 - x_3 - x_2$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Update:
$x_4 := x_4 + 0.5$

Simplex: Example (cnt'd)

$$x_5 = x_2 - x_1$$
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Update:
$x_4 := x_4 + 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Simplex: Example (cnt'd)

$$x_5 = x_2 - x_1$$
$$x_6 = x_3 + x_1$$
$$x_7 = x_4 - x_3 - x_2$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Simplex: Example (cnt'd)

$$x_5 = x_2 - x_1$$
$$x_6 = x_3 + x_1$$
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Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

$$x_5 = x_2 - x_1$$
$$x_6 = x_3 + x_1$$
$$x_7 = x_4 - x_3 - x_2$$

Pivot: x_7, x_2

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Simplex: Example (cnt'd)

$$x_5 = x_2 - x_1$$

$$x_6 = x_3 + x_1$$

$$x_7 = x_4 - x_3 - x_2 \qquad \leftarrow \qquad x_2 = x_4 - x_3 - x_7$$

Pivot: x_7, x_2

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Simplex: Example (cnt'd)

$$x_5 = x_2 - x_1 \qquad \leftarrow \qquad x_5 = x_4 - x_3 - x_7 - x_1$$

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0	x_1	0	1
	x_2	0	
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0	x_5	0	0
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$$x_5 = x_4 - x_3 - x_7 - x_1$$
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0	x_1	0	1
	x_2	0	
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$$x_5 = x_4 - x_3 - x_7 - x_1$$
$$x_6 = x_3 + x_1$$
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Update:
$x_7 := x_7 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Simplex: Example (cnt'd)

$$x_5 = x_4 - x_3 - x_7 - x_1$$

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Update:

$x_7 := x_7 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0	
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0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Simplex: Example (cnt'd)

$$x_5 = x_4 - x_3 - x_7 - x_1$$
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Update:
$x_7 := x_7 - 0.5$

Lower B.	Var	Value	Upper B.
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	x_3	0	
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0	x_5	0.5	0
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Simplex: Example (cnt'd)

$$x_5 = x_4 - x_3 - x_7 - x_1$$

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Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0.5	
	x_3	0	
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Pivot: x_5, x_1

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0	x_1	0	1
	x_2	0.5	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Simplex: Example (cnt'd)

$x_5 = x_4 - x_3 - x_7 - x_1 \qquad \leftarrow \qquad x_1 = x_4 - x_3 - x_7 - x_5$

$x_6 = x_3 + x_1$

$x_2 = x_4 - x_3 - x_7$

Pivot: x_5, x_1

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0.5	
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0.5	x_4	0.5	1
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0	x_6	0	0
0	x_7	0	0

Simplex: Example (cnt'd)

$x_5 = x_4 - x_3 - x_7 - x_1 \quad \leftarrow \quad x_1 = x_4 - x_3 - x_7 - x_5$

$x_6 = x_3 + x_1 \quad \leftarrow \quad x_6 = x_4 - x_7 - x_5$

$x_2 = x_4 - x_3 - x_7$

Pivot: x_5, x_1

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0.5	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Simplex: Example (cnt'd)

$$x_1 = x_4 - x_3 - x_7 - x_5$$
$$x_6 = x_4 - x_7 - x_5$$
$$x_2 = x_4 - x_3 - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0.5	
	x_3	0	
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$$x_1 = x_4 - x_3 - x_7 - x_5$$

$$x_6 = x_4 - x_7 - x_5$$

$$x_2 = x_4 - x_3 - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0.5	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Update:

$x_5 := x_5 - 0.5$

Simplex: Example (cnt'd)

$$x_1 = x_4 - x_3 - x_7 - x_5$$

$$x_6 = x_4 - x_7 - x_5$$

$$x_2 = x_4 - x_3 - x_7$$

Update:

$x_5 := x_5 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2	0.5	
	x_3	0	
0.5	x_4	0.5	1
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0	x_1	0.5	1
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0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Update:
$x_5 := x_5 - 0.5$

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$$x_1 = x_4 - x_3 - x_7 - x_5$$
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$$x_1 = x_4 - x_3 - x_7 - x_5$$

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Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
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	x_3	0	
0.5	x_4	0.5	1
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Simplex: Example (cnt'd)

$$x_1 = x_4 - x_3 - x_7 - x_5$$
$$x_6 = x_4 - x_7 - x_5$$
$$x_2 = x_4 - x_3 - x_7$$

Failure

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2	0.5	
	x_3	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

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- For notation:

$$\text{slack}^+(x_i) = \{x_j \notin \mathcal{B} \mid (T_{i,j} > 0 \wedge \alpha(x_j) < u(x_j)) \vee (T_{i,j} < 0 \wedge \alpha(x_j) > l(x_j))$$
$$\text{slack}^-(x_i) = \{x_j \notin \mathcal{B} \mid (T_{i,j} < 0 \wedge \alpha(x_j) < u(x_j)) \vee (T_{i,j} > 0 \wedge \alpha(x_j) > l(x_j))$$

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$$\text{Pivot}_1 \quad \frac{x_i \in \mathcal{B}, \quad \alpha(x_i) < l(x_i), \quad x_j \in \mathsf{slack}^+(x_i)}{T := \textit{pivot}(T, i, j), \quad \mathcal{B} := \mathcal{B} \cup \{x_j\} \setminus \{x_i\}}$$

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$$\text{Update} \quad \frac{x_j \notin \mathcal{B}, \quad \alpha(x_j) < l(x_j) \vee \alpha(x_j) > u(x_j), \quad l(x_j) \leq \alpha(x_j) + \delta \leq u(x_j)}{\alpha := \textit{update}(\alpha, x_j, \delta)}$$

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$$\text{Update} \quad \frac{x_j \notin \mathcal{B}, \quad \alpha(x_j) < l(x_j) \lor \alpha(x_j) > u(x_j), \quad l(x_j) \le \alpha(x_j) + \delta \le u(x_j)}{\alpha := \textit{update}(\alpha, x_j, \delta)}$$

$$\text{Failure} \quad \frac{x_i \in \mathcal{B}, \quad (\alpha(x_i) < l(x_i) \ \land \ \mathsf{slack}^+(x_i) = \emptyset) \lor (\alpha(x_i) > u(x_i) \ \land \ \mathsf{slack}^-(x_i) = \emptyset)}{\texttt{UNSAT}}$$

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$$\text{Success} \quad \frac{\forall x_i \in \mathcal{X}. \ l(x_i) \leq \alpha(x_i) \leq u(x_i)}{\texttt{SAT}}$$

Properties of Simplex

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Theorem (Soundness and Completeness of Simplex)

*The simplex algorithm is sound and complete**

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  - But unfortunately quite slow
- Better selection strategies exist (e.g., *steepest edge*)
- Problem is in $\mathbf{P}$, unknown whether simplex is in $\mathbf{P}$

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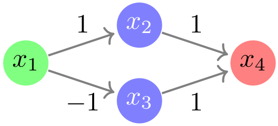
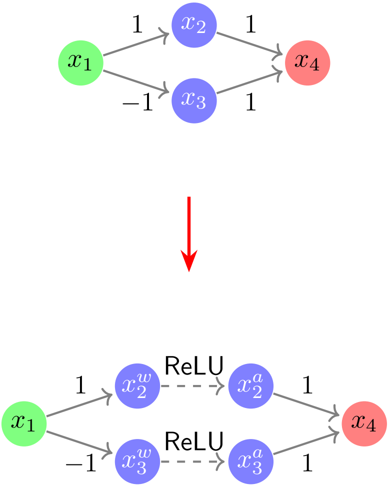
  - May violate ReLU constraint
  - Similar to bound constraints
  - Fix *incrementally*
- Use pivots and updates, same as before

Reluplex: Example

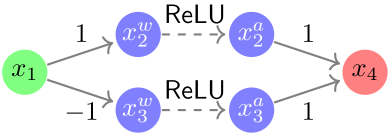


Reluplex: Example

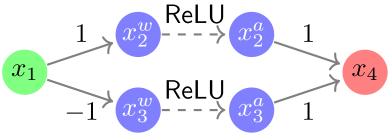


Reluplex: Example (cnt'd)

Reluplex: Example (cnt'd)

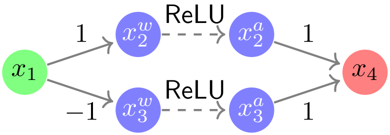


Reluplex: Example (cnt'd)



- Equations for weighted sums:

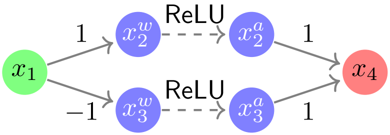
Reluplex: Example (cnt'd)



- Equations for weighted sums:

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Reluplex: Example (cnt'd)

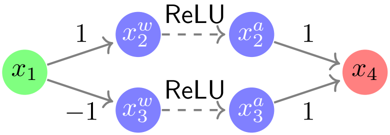


- Equations for weighted sums:

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

- Bounds:

Reluplex: Example (cnt'd)



- Equations for weighted sums:

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

- Bounds:

$$x_1 \in [0, 1]$$
$$x_4 \in [0.5, 1]$$
$$x_2^w, x_3^w \text{ unbounded}$$
$$x_2^a, x_3^a \in [0, \infty)$$
$$x_5, x_6, x_7 \in [0, 0]$$

Reluplex: Example (cnt'd)

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Update:

$x_4 := x_4 + 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Update:

$x_4 := x_4 + 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Update:
$x_4 := x_4 + 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_7 = x_4 - x_3^a - x_2^a$$

Pivot: x_7, x_2^a

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$x_5 = x_2^w - x_1$

$x_6 = x_3^w + x_1$

$x_7 = x_4 - x_3^a - x_2^a$

Pivot: x_7, x_2^a

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Pivot: x_7, x_2^a

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:
$x_7 := x_7 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$x_5 = x_2^w - x_1$

$x_6 = x_3^w + x_1$

$x_2^a = x_4 - x_3^a - x_7$

Update:

$x_7 := x_7 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0.5	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:

$x_7 := x_7 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Update:
$x_2^w := x_2^w + 0.5$

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0	0
0	x_7	0	0

Update:
$x_2^w := x_2^w + 0.5$

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Update:
$x_2^w := x_2^w + 0.5$

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Pivot: x_5, x_1

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_5 = x_2^w - x_1$$
$$x_6 = x_3^w + x_1$$
$$x_2^a = x_4 - x_3^a - x_7$$

Pivot: x_5, x_1

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Pivot: x_5, x_1

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:
$x_5 := x_5 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:
$x_5 := x_5 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0.5	0
0	x_6	0	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:

$x_5 := x_5 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Pivot: x_6, x_3^w

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_6 = x_3^w + x_2^w - x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Pivot: x_6, x_3^w

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_3^w = x_6 - x_2^w + x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_3^w = x_6 - x_2^w + x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:

$x_6 := x_6 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_3^w = x_6 - x_2^w + x_5$$
$$x_2^a = x_4 - x_3^a - x_7$$

Update:

$x_6 := x_6 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	0	
0	x_3^a	0	
0.5	x_4	0.5	1
0	x_5	0	0
0	x_6	0.5	0
0	x_7	0	0

Reluplex: Example (cnt'd)

$$x_1 = x_2^w - x_5$$
$$x_3^w = x_6 - x_2^w + x_5$$
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Update:

$x_6 := x_6 - 0.5$

Lower B.	Var	Value	Upper B.
0	x_1	0.5	1
	x_2^w	0.5	
0	x_2^a	0.5	
	x_3^w	-0.5	
0	x_3^a	0	
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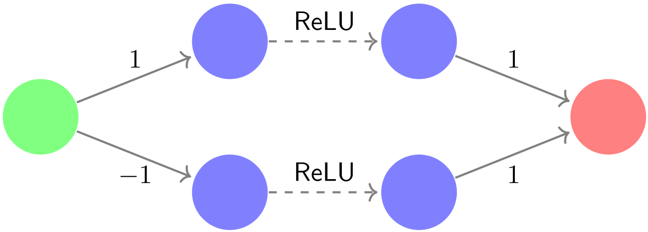

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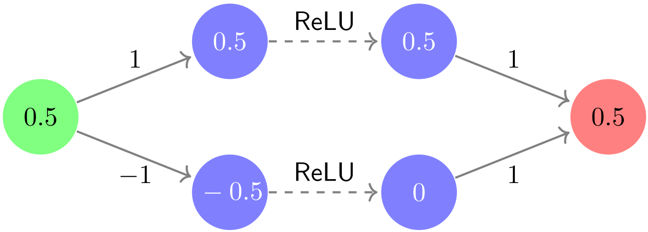

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    - $R \subset \mathcal{X} \times \mathcal{X}$ is a set of ReLU connections

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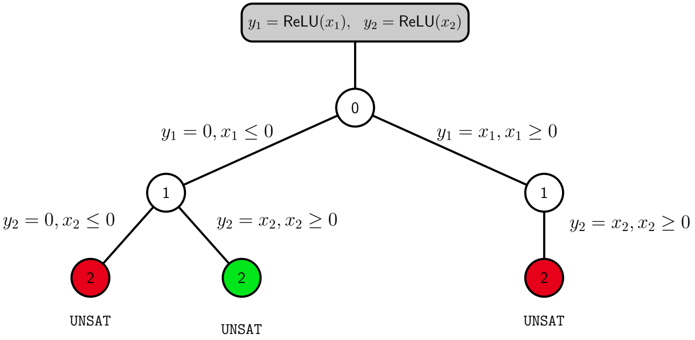

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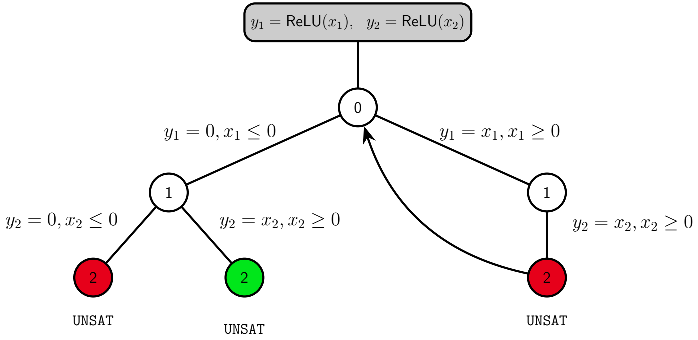

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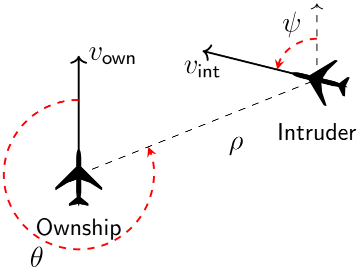

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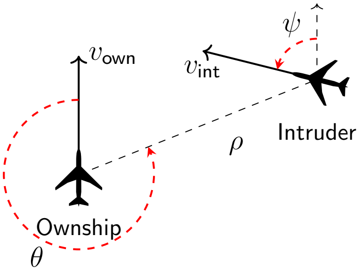

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Certifying ACAS Xu

- There are properties that the FAA cares about
  - Consistent alerting regions
  - No unnecessary turning advisories
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- Properties defined formally
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Certifying ACAS Xu (cnt'd)

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Certifying ACAS Xu (cnt'd)

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Certifying ACAS Xu (cnt'd)

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	Networks	Result	Time	Stack	Splits
ϕ_1	41	UNSAT	394517	47	1522384
	4	TIMEOUT			
ϕ_2	1	UNSAT	463	55	88388
	35	SAT	82419	44	284515
ϕ_3	42	UNSAT	28156	22	52080
ϕ_4	42	UNSAT	12475	21	23940
ϕ_5	1	UNSAT	19355	46	58914
ϕ_6	1	UNSAT	180288	50	548496
ϕ_7	1	TIMEOUT			
ϕ_8	1	SAT	40102	69	116697
ϕ_9	1	UNSAT	99634	48	227002
ϕ_{10}	1	UNSAT	19944	49	88520

Adversarial Robustness

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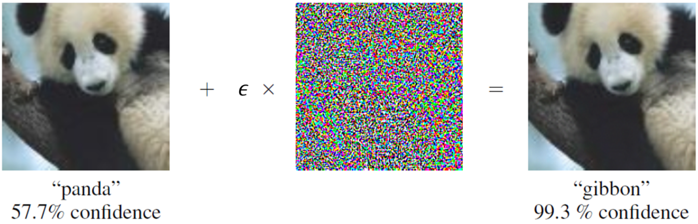
Goodfellow et al., 2015



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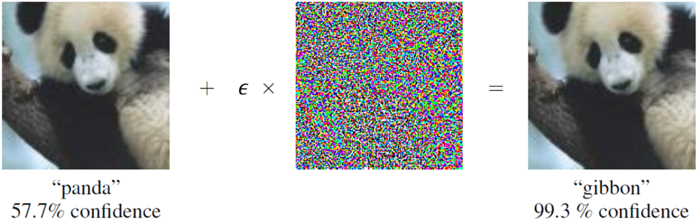

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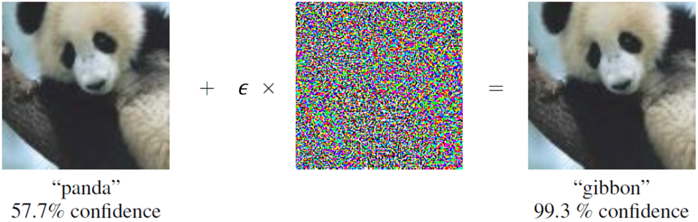

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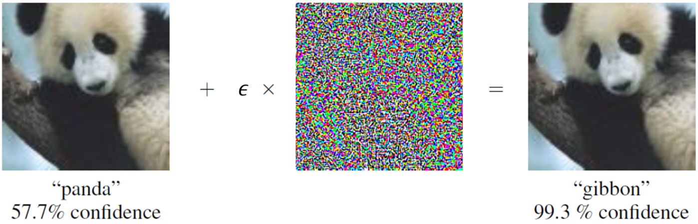

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- Verification can prove that this cannot occur
- Allows us to assess attacks defenses

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	Result	Time	Result	Time	Result	Time	Result	Time	Result	Time
Point 1	SAT	135	SAT	239	SAT	24	UNSAT	609	UNSAT	57
Point 2	UNSAT	5880	UNSAT	1167	UNSAT	285	UNSAT	57	UNSAT	5
Point 3	UNSAT	863	UNSAT	436	UNSAT	99	UNSAT	53	UNSAT	1
Point 4	SAT	2	SAT	977	SAT	1168	UNSAT	656	UNSAT	7
Point 5	UNSAT	14560	UNSAT	4344	UNSAT	1331	UNSAT	221	UNSAT	6

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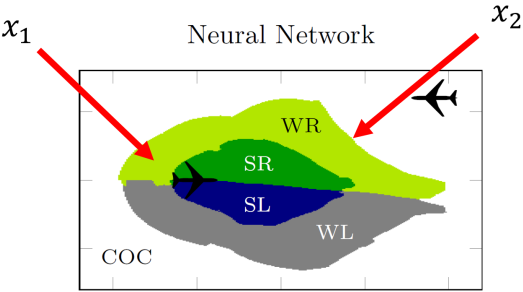

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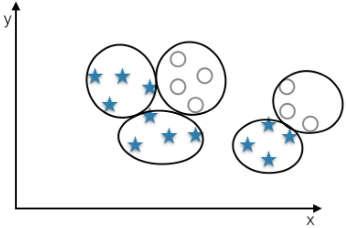

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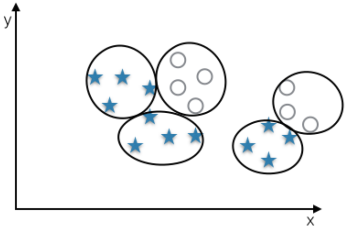

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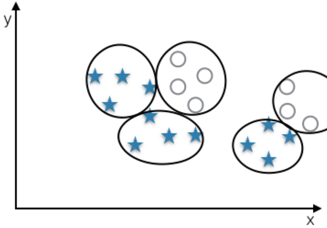

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Thank You!

Questions

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