# Maximizing Revenue in Expectation

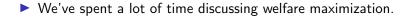
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#### Overview

- ▶ We've spent a lot of time discussing welfare maximization.
- But many auctions have a more pecuniary goal. What if we want to maximize revenue?

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- What does that mean? What is our benchmark?
- This lecture: a case study for single item auctions.



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- What about for revenue? Not so simple.
- Consider a single bidder, single item auction. Offering a fixed price p is always dominant strategy truthful.
- ▶ Revenue is p if  $v_i \ge p$ , 0 otherwise.
- So ex-post, the revenue-optimal auction sets p = v<sub>i</sub>... But ex-ante, we don't have enough information.

Suppose we know that bidders have valuations v<sub>i</sub> ~ D for some distribution D.

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Where  $F(p) = \Pr_{v \sim D}[v \leq p]$ .

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Where  $F(p) = \Pr_{v \sim D}[v \leq p]$ . • E.g. if D is uniform on [0, 1], then F(p) = p and:

$$\max_{p} Rev(p) = \frac{1}{2} \cdot (1 - \frac{1}{2}) = \frac{1}{4}$$

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We want to design a truthful mechanism (X, P) that maximizes:

$$\mathbf{E}_{v\sim D^n}\left[\sum_{i=1}^n P_i(v)\right]$$

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And we know:

$$P_i(v) = v_i \cdot X_i(v) - \int_0^{v_i} X_i(z, v_{-i}) dz$$

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- Plan: Find X to maximize:

$$\mathbf{E}_{\mathbf{v}\sim D^n}\left[\sum_{i=1}^n P_i(\mathbf{v})\right] = \sum_{i=1}^n \mathbf{E}_{\mathbf{v}_{-i}\sim D^{n-1}}\left[\mathbf{E}_{\mathbf{v}_i\sim D}\left[P_i(\mathbf{v}_i,\mathbf{v}_{-i})\right]\right]$$

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• Notation: f(p) is the pdf of D.

$$F(p) = \Pr_{v \sim D}[v \le p] = \int_0^p f(v) dv$$

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Consider the inner term:

$$\operatorname{E}_{\mathbf{v}_i}\left[P_i(\mathbf{v})\right] = \operatorname{E}_{\mathbf{v}_i}\left[v_i \cdot X_i(v_i, v_{-i}) - \int_0^{v_i} X_i(z, v_{-i}) dz\right]$$

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So: We want to maximize

$$\mathbf{E}_{\mathbf{v}\sim D^n}\left[\sum_{i=1}^n \phi(\mathbf{v}_i) \cdot X(\mathbf{v})\right] \qquad \underbrace{\phi(\mathbf{v}_i) = \left(\mathbf{v}_i - \frac{(1 - F(\mathbf{v}_i))}{f(\mathbf{v}_i)}\right)}_{\text{"Virtual Value"}}$$

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- Our objective looks just like welfare with values replaced by virtual values.
- ► (Pointwise) optimal allocation rule: Give the item to the bidder *i* with highest φ(v<sub>i</sub>) if it's positive. Otherwise give the item to nobody.
- This is a monotone allocation rule if *D* is *regular*:  $\phi(v_i)$  is monotone.
  - e.g. if D is uniform,  $\phi(v_i) = v_i (1 v_i) = 2v_i 1$
  - Note that \(\phi^{-1}(0)\) recovers the optimal \(p = 1/2\) for a single bidder.

What do revenue maximizing auctions look like? (when  $v_i$  drawn iid from regular D)

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- ▶ Because \(\phi\) is monotone, \(i^\* = \arg max\_i v\_i: the item goes to the highest bidder when \(\phi(v\_{i^\*}) \ge 0.\)

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- Because φ is monotone, i<sup>\*</sup> = arg max<sub>i</sub> v<sub>i</sub>: the item goes to the highest bidder when φ(v<sub>i</sub>\*) ≥ 0.
- Winner pays  $v_{i^*} \int_{p^*}^{v_{i^*}} 1 = p^*$ , where:

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▶ i.e. its just a Vickrey auction with a reserve price of φ<sup>-1</sup>(0)!
 ▶ Remarkable — Simple eBay style auction is *the best possible*.

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  - Each bidder has their own virtual valuation function  $\phi_i(v_i)$ .

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- Doesn't extend beyond single parameter domains...

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- Auction no longer so natural. e.g. high bidder no longer necessarily wins.
- Doesn't extend beyond single parameter domains...
- Requires knowledge of D...

If we care about revenue, should we give up on welfare?

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So we might be better off maximizing welfare with more bidders...

#### Theorem

Consider bidders drawn i.i.d. from a regular distribution D. For any  $n \ge 1$ , the Vickrey auction with n + 1 bidders has higher expected revenue than the revenue optimal auction with n bidders.

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Consider bidders drawn i.i.d. from a regular distribution D. For any  $n \ge 1$ , the Vickrey auction with n + 1 bidders has higher expected revenue than the revenue optimal auction with n bidders. So recruiting just one extra bidder is worth more than optimizing revenue for the current population.

Consider the hypothetical auction A for n + 1 bidders:

1. Run the revenue optimal auction for the first n bidders.

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Consider the hypothetical auction A for n + 1 bidders:

- 1. Run the revenue optimal auction for the first n bidders.
- 2. If the auction fails to allocate the item, give it to bidder n + 1 for free.

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#### **Observations**:

- 1. The revenue of A is exactly equal to the optimal revenue obtainable from n bidders.
- 2. A always allocates the item.

Claim: The Vickrey mechanism is obtains the maximum revenue amongst all mechanisms that always allocate the item.

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• Recall that  $E_{v}[\sum_{i} P_{i}(v)] = E[\sum_{i} \phi_{i}(v_{i}) \cdot X_{i}(v)].$ 

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- Recall that  $E_{v}[\sum_{i} P_{i}(v)] = E[\sum_{i} \phi_{i}(v_{i}) \cdot X_{i}(v)].$
- We can maximize the RHS (subject to always allocating the item) by always allocating to arg max<sub>i</sub> φ(v<sub>i</sub>).

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- Since D is regular, \u03c6 is monotone: this is arg max<sub>i</sub> v<sub>i</sub> the Vickrey allocation!
- So: The Vickrey-auction with n + 1 bidders has only higher revenue than the optimal n bidder auction.

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### Thanks!

See you next class — stay healthy!

