Stable Matchings

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- We will again prohibit the use of money...
- Used in practice to match medical students to residencies, pledges to sororities, students to public schools in various districts.

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A *matching* $\mu : M \cup W \to M \cup W$ is an assignment of students to schools so that each student is assigned to exactly one school and vice versa. For each $m \in M$ and $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

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3. Each $m \in M$ has a strict preference ordering $\succ_m$ over the set W, and each $w \in W$ has a strict preference ordering $\succ_w$ over the set M.

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- Just as in last lecture, we have two desiderate:
  1. We would like the matching that we compute to be *good* in some sense, and
  2. We would like to incentivize participants to reveal their true preferences to the mechanism.
- We'll be able to find "good" matchings — and will have limited success managing preferences.

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$$w \succ_m \mu(m) \quad \text{and} \quad m \succ_w \mu(w)$$

We call such an (m, w) pair a *blocking pair* for μ. (A blocking pair witnesses instability because m and w could mutually benefit by leaving their proposed partners and pairing with one another).
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3. We might more ambitiously want to compute the "best" stable matching – but do they even exist?

They Do Exist!

Theorem (Gale and Shapley)

For any set of preferences $(\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n})$*, a stable matching* μ *exists.*

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2. The student applying *deferred acceptance* algorithm.

Algorithm 1 The Deferred Acceptance Algorithm (Student Applying Version)

DeferredAcceptance($\succ$):

Initially, $\mu(m) = \emptyset$ for all $m \in M$. (i.e. nobody is yet matched).
Each student $m \in M$ *applies* to his most preferred $w \in W$. For each school $w \in W$, let m' be its most preferred student among the set that applied to it, and set $\mu(m') \leftarrow w$. All other students are *rejected* (and hence unmatched).
while There exists any unmatched student $m \in M$: **do**
m **applies** to his most preferred $w \in W$ that he has not yet applied to.
If $m \succ_w \mu(w)$, then $\mu(\mu(w)) \leftarrow \emptyset$ and $\mu(w) \leftarrow m$ (i.e. w rejects its current match and instead matches to m). **Else**, m is rejected.
end while
Return μ

Proof

1. The algorithm halts: every school receives at least one application over the course of the algorithm. (If there is a school without an application, there is an unmatched student, and the algorithm has not halted unless he has applied to all schools). Once a school has received an application, it becomes matched, and stays matched for the rest of the algorithm.

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2. Since $|W| = |M|$, once all schools are matched, all students are matched.
3. So the algorithm halts after at most n^2 applications, since no student applies to the same school twice.

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5. Since schools only ever change who they are matched to in favor of more preferred students, we must have:

$$\mu(w_1) \succeq_{w_1} m' \succ_{w_1} m_1$$

which contradicts $m_1 \succ_{w_1} \mu(w_1)$.

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6. Tada!

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3. Optimality: The best among all achievable matchings:

Definition

A matching μ is *student optimal* if for every achievable pair (m, w), $\mu(m) \succeq_m w$ Similarly, we can define *school optimal* matchings, and student and school *pessimal* matchings. (A matching μ is school pessimal if for every achievable pair (m, w), $m \succeq_w \mu(w)$)

Its Good to be on the Applying Side

Theorem

The stable matching μ output by the student-applying deferred acceptance algorithm is student optimal.

Proof

1. Suppose otherwise. There must be some first round k at which a student m is rejected by his most preferred achievable school w, in favor of m'. $m' \succ_w m$.

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Its Bad to be on the Receiving Side

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The stable matching produced by the student-applying deferred acceptance algorithm is school pessimal.

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What about Incentives?

Theorem

The student applying deferred acceptance algorithm is dominant strategy incentive compatible for the students. (i.e. reporting their true preferences $\succ_m$ *is a dominant strategy for each* $m \in M$*).*

Proof

1. Suppose otherwise: there is a set of preferences $\succ = (\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n})$ and a deviation $\succ'_{m_1}$ such that if $\mu = DE(\succ)$ and $\mu' = DE(\succ')$ (where $\succ' = (\succ'_{m_1}, \succ_{-m_1})$), then:

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2. We know that μ is stable and student optimal with respect to preferences $\succ$, and μ' is stable and student optimal with respect to preferences $\succ'$

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   3.2 The set of schools whose matches in μ' are in R (and so prefer them to their match in μ):
   $$T = \{w : \mu'(w) \in R\}$$

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2. We'll start with the first claim...

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5. Because μ' is stable w.r.t. $\succ'$, it must be that $\mu'(m') \succ_{m'} \mu(m') = w$.

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6. Hence $m' \in R$ as we wanted

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3. Let m_ℓ be the *last* $m \in R$ who applies during the DE algorithm. This application must be to $\mu(m_\ell) \equiv w_\ell$.
4. By the first claim, since $m_\ell \in R$, $w_\ell \in T$.
5. It must be that w_ℓ rejected $\mu'(w_\ell)$ at a strictly earlier round (since m_ℓ is the last $m \in R$ to apply), and hence when m_ℓ applies to w_ℓ, w_ℓ rejects some $m_r \notin R$ such that: $m_r \succ_{w_\ell} \mu'(w_\ell)$

Proof

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4. Tada!

Thanks!

See you next class — stay healthy!