Stable Matchings

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Used in practice to match medical students to residencies, pledges to sororities, students to public schools in various districts.
A Model

1. Let $M$ and $W$ denote sets of students and schools respectively. Assume $|M| = |W| = n$. 

2. A Matching: Definition

A matching $\mu: M \cup W \rightarrow M \cup W$ is an assignment of students to schools so that each student is assigned to exactly one school and vice versa. For each $m \in M$ and $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

3. Each $m \in M$ has a strict preference ordering $\succ_m$ over the set $W$, and each $w \in W$ has a strict preference ordering $\succ_w$ over the set $M$. 
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Goals

Just as in last lecture, we have two desiderate:

1. We would like the matching that we compute to be good in some sense, and
2. We would like to incentivize participants to reveal their true preferences to the mechanism.

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What makes a Matching Reasonable

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2. An equilibrium like condition:

Definition
A matching \( \mu \) is *unstable* if there exists an \( m \in M \) and \( w \in W \) such that \( \mu(m) \neq w \), but:

\[
w \succ_m \mu(m) \quad \text{and} \quad m \succ_w \mu(w)
\]

We call such an \((m, w)\) pair a *blocking pair* for \( \mu \). (A blocking pair witnesses instability because \( m \) and \( w \) could mutually benefit by leaving their proposed partners and pairing with one another).

A matching \( \mu \) is *stable* if it has no blocking pairs.
What makes a Matching Reasonable

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We call such an $(m, w)$ pair a *blocking pair* for $\mu$. (A blocking pair witnesses instability because $m$ and $w$ could mutually benefit by leaving their proposed partners and pairing with one another).

A matching $\mu$ is *stable* if it has no blocking pairs.

3. We might more ambitiously want to compute the “best” stable matching – but do they even exist?
Theorem (Gale and Shapley)

For any set of preferences \((\succ m_1, \ldots, \succ m_n, \succ w_1, \ldots, \succ w_n)\), a stable matching \(\mu\) exists.
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For any set of preferences $(\succsim_{m_1}, \ldots, \succsim_{m_n}, \succsim_{w_1}, \ldots, \succsim_{w_n})$, a stable matching $\mu$ exists.

1. An algorithmic proof: we’ll prove existence by showing how to find one.
They Do Exist!

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For any set of preferences \((\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n})\), a stable matching \(\mu\) exists.

1. An algorithmic proof: we’ll prove existence by showing how to find one.
2. The student applying deferred acceptance algorithm.
Algorithm 1 The Deferred Acceptance Algorithm (Student Applying Version)

DeferredAcceptance(≻):

Initially, \( \mu(m) = \emptyset \) for all \( m \in M \). (i.e. nobody is yet matched).
Each student \( m \in M \) applies to his most preferred \( w \in W \). For each school \( w \in W \), let \( m' \) be its most preferred student among the set that applied to it, and set \( \mu(m') \leftarrow w \). All other students are rejected (and hence unmatched).

while There exists any unmatched student \( m \in M \): do

  \( m \) applies to his most preferred \( w \in W \) that he has not yet applied to.

  If \( m \succ_w \mu(w) \), then \( \mu(\mu(w)) \leftarrow \emptyset \) and \( \mu(w) \leftarrow m \) (i.e. \( w \) rejects its current match and instead matches to \( m \)). Else, \( m \) is rejected.

end while

Return \( \mu \)
Proof

1. The algorithm halts: every school receives at least one application over the course of the algorithm. (If there is a school without an application, there is an unmatched student, and the algorithm has not halted unless he has applied to all schools). Once a school has received an application, it becomes matched, and stays matched for the rest of the algorithm.
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2. Since $|W| = |M|$, once all schools are matched, all students are matched.
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3. So the algorithm halts after at most $n^2$ applications, since no student applies to the same school twice.
Proof

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2. Suppose otherwise: there is a blocking pair $(m_1, w_1)$ with $\mu(m_1) \neq w_1$, but $w_1 \succ_{m_1} \mu(m_1)$ and $m_1 \succ_{w_1} \mu(w_1)$. 

5. Since schools only ever change who they are matched to in favor of more preferred students, we must have: $\mu(w_1) \succeq w_1 m'_1 \succ w_1 m_1$ which contradicts $m_1 \succ w_1 \mu(w_1)$. 

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For \( m \in M \) and \( w \in W \), we say that \( w \) is achievable for \( m \) (and vice versa) if there exists a stable matching \( \mu \) such that \( \mu(m) = w \).

3. Optimality: The best among all achievable matchings:

Definition
A matching \( \mu \) is student optimal if for every achievable pair \( (m, w) \), \( \mu(m) \succeq_m w \). Similarly, we can define school optimal matchings, and student and school pessimal matchings. (A matching \( \mu \) is school pessimal if for every achievable pair \( (m, w) \), \( m \succeq_w \mu(w) \))
Its Good to be on the Applying Side

Theorem

The stable matching $\mu$ output by the student-applying deferred acceptance algorithm is student optimal.
Proof

1. Suppose otherwise. There must be some first round $k$ at which a student $m$ is rejected by his most preferred achievable school $w$, in favor of $m'$. $m' \succ_w m$.

2. Since $w$ is achievable for $m$, there must be some stable matching $\mu$ such that $\mu(m) = w$ and $\mu(m') = w'$ (and hence $w'$ is achievable for $m'$).

3. We must have $w \succ m' w'$ (since $m'$ applied to $w$, and can't have been rejected by any achievable school since by assumption, $k$ was the first round at which a student was rejected by an achievable school.)

4. Combining: $m' \succ w m w' \succ m w'$.

5. $(m', w)$ form a blocking pair for $\mu$, contradicting stability.

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Proof

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2. There exists some $w$ with $\mu(w) = m$, and $m \succ_w m'$ for some other achievable student $m'$.

3. So there must exist a different stable matching $\mu'$ with $\mu'(m') = w$, and $\mu'(m) = w'$.

4. But we must have $w \succ m w' = \mu'(m)$ because $\mu$ is student-optimal and $w'$ is achievable for $m'$.

5. So $(m, w)$ are a blocking pair for $\mu'$, which contradicts its stability.

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What about Incentives?

Theorem

The student applying deferred acceptance algorithm is dominant strategy incentive compatible for the students. (i.e. reporting their true preferences $\succ_m$ is a dominant strategy for each $m \in M$).
Proof

1. Suppose otherwise: there is a set of preferences 
\( \succeq = (\succeq_{m_1}, \ldots, \succeq_{m_n}, \succeq_{w_1}, \ldots, \succeq_{w_n}) \) and a deviation \( \succeq'_{m_1} \) such that if \( \mu = DE(\succeq) \) and \( \mu' = DE(\succeq') \) (where 
\( \succeq' = (\succeq'_{m_1}, \succeq_{-m_1}) \)), then:

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\mu'(m_1) \succ_m \mu(m_1).
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Proof

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2. We know that \(\mu\) is stable and student optimal with respect to preferences \(\succ\), and \(\mu'\) is stable and student optimal with respect to preferences \(\succ'\).
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3. Define two sets:

\[R = \{m : \mu'(m) \succ m \mu(m)\}\]

\[T = \{w : \mu'(w) \in R\}\]
Proof

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   3.1 The set of students who prefer $\mu'$ to $\mu$:

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3.1 The set of students who prefer $\mu'$ to $\mu$:

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3.2 The set of schools whose matches in $\mu'$ are in $R$ (and so prefer them to their match in $\mu$):

$$T = \{w : \mu'(w) \in R\}$$
Proof

1. We will show:

   \( w \in T \iff \mu(w) \in R \) (i.e. if a school's partner in \( \mu' \) prefers \( \mu' \) to \( \mu \), so does its partner in \( \mu \)), and from this derive that:

   1.2 There exists a \( w_\ell \in T \) and a \( m_r \in R \) such that \((w_\ell, m_r)\) form a blocking pair in \( \mu' \) with respect to \( \succ' \), a contradiction.

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3. Since \( m \in R \), we know that: \( w = \mu'(m) \succ_m \mu(m) \).
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3. Since \( m \in R \), we know that: \( w = \mu'(m) \succ_m \mu(m) \).

4. Since \( \mu \) is stable w.r.t \( \succ \), it must be that \( \mu(w) = m' \succ_w m \).

5. Because \( \mu' \) is stable w.r.t. \( \succ' \), it must be that \( \mu'(m') \succ_{m'} \mu(m') = w \).
Proof

Claim

\[ w \in T \iff \mu(w) \in R \]

1. For any \( m \in R \), let \( w = \mu'(m) \in T \). Let \( m' = \mu(w) \) be \( w \)'s partner in \( \mu \).

2. If \( m' = m_1 \), we are done. Otherwise we can assume \( m' \neq m_1 \), and therefore that \( \succ_m = \succ'_m \).

3. Since \( m \in R \), we know that: \( w = \mu'(m) \succ_m \mu(m) \).

4. Since \( \mu \) is stable w.r.t \( \succ \), it must be that \( \mu(w) = m' \succ_w m \).

5. Because \( \mu' \) is stable w.r.t. \( \succ' \), it must be that \( \mu'(m') \succ_{m'} \mu(m') = w \).

6. Hence \( m' \in R \) as we wanted
Proof

Claim
There exists a \( w_\ell \in T \) and a \( m_r \in R \) such that \((w_\ell, m_r)\) form a blocking pair in \( \mu' \) with respect to \( \succ' \).

2. So when running \( \text{DE}(\succ) \), it must be that every \( m \in R \) applies to \( \mu'(m) \), and is rejected by \( \mu'(m) \) at some round.

3. Let \( m_\ell \) be the last \( m \in R \) who applies during the \( \text{DE} \) algorithm. This application must be to \( \mu(m_\ell) \equiv w_\ell \).

4. By the first claim, since \( m_\ell \in R \), \( w_\ell \in T \).

5. It must be that \( w_\ell \) rejected \( \mu'(w_\ell) \) at a strictly earlier round (since \( m_\ell \) is the last \( m \in R \) to apply), and hence when \( m_\ell \) applies to \( w_\ell \), \( w_\ell \) rejects some \( m_r \not\in R \) such that:

\[ m_r \succ w_\ell \mu'(w_\ell) \]
Proof

Claim
There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$. 

1. Since for every $m \in R$, $\mu'(m) \succ_m \mu(m)$, by stability, it must be that for all $w \in T$: $\mu(w) \succ_w \mu'(w)$. 

2. So when running DE($\succ'$), it must be that every $m \in R$ applies to $\mu'(m)$, and is rejected by $\mu'(m)$ at some round.

3. Let $m_\ell$ be the last $m \in R$ who applies during the DE algorithm. This application must be to $\mu(m_\ell) \equiv w_\ell$.

4. By the first claim, since $m_\ell \in R$, $w_\ell \in T$.

5. It must be that $w_\ell$ rejected $\mu'(w_\ell)$ at a strictly earlier round (since $m_\ell$ is the last $m \in R$ to apply), and hence when $m_\ell$ applies to $w_\ell$, $w_\ell$ rejects some $m_r \not\in R$ such that: $m_r \succ w_\ell \mu'(w_\ell)$.
Proof

Claim
There exists a \( w_\ell \in T \) and a \( m_r \in R \) such that \((w_\ell, m_r)\) form a blocking pair in \( \mu' \) with respect to \( \succ' \)

1. Since for every \( m \in R, \mu'(m) \succ_m \mu(m) \), by stability, it must be that for all \( w \in T \): \( \mu(w) \succ_w \mu'(w) \).

2. So when running \( \text{DE}(\succ') \), it must be that every \( m \in R \) applies to \( \mu'(m) \), and is rejected by \( \mu'(m) \) at some round.
Proof

Claim
There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$

1. Since for every $m \in R$, $\mu'(m) \succ_m \mu(m)$, by stability, it must be that for all $w \in T$: $\mu(w) \succ_w \mu'(w)$.

2. So when running $\text{DE}(\succ)$, it must be that every $m \in R$ applies to $\mu'(m)$, and is rejected by $\mu'(m)$ at some round.

3. Let $m_\ell$ be the last $m \in R$ who applies during the DE algorithm. This application must be to $\mu(m_\ell) \equiv w_\ell$. 
Proof

Claim
There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$

1. Since for every $m \in R$, $\mu'(m) \succ_m \mu(m)$, by stability, it must be that for all $w \in T$: $\mu(w) \succ_w \mu'(w)$.

2. So when running $\text{DE}(\succ)$, it must be that every $m \in R$ applies to $\mu'(m)$, and is rejected by $\mu'(m)$ at some round.

3. Let $m_\ell$ be the last $m \in R$ who applies during the $\text{DE}$ algorithm. This application must be to $\mu(m_\ell) \equiv w_\ell$.

4. By the first claim, since $m_\ell \in R$, $w_\ell \in T$. 
Proof

Claim

There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$

1. Since for every $m \in R$, $\mu'(m) \succ_m \mu(m)$, by stability, it must be that for all $w \in T$: $\mu(w) \succ_w \mu'(w)$.

2. So when running DE($\succ$), it must be that every $m \in R$ applies to $\mu'(m)$, and is rejected by $\mu'(m)$ at some round.

3. Let $m_\ell$ be the last $m \in R$ who applies during the DE algorithm. This application must be to $\mu(m_\ell) \equiv w_\ell$.

4. By the first claim, since $m_\ell \in R$, $w_\ell \in T$.

5. It must be that $w_\ell$ rejected $\mu'(w_\ell)$ at a strictly earlier round (since $m_\ell$ is the last $m \in R$ to apply), and hence when $m_\ell$ applies to $w_\ell$, $w_\ell$ rejects some $m_r \not\in R$ such that: $m_r \succ_{w_\ell} \mu'(w_\ell)$
Proof

\[ m_r \succeq_{w_\ell} \mu'(w_\ell) \]

1. Since \( m_r \) had applied to \( w_\ell \) before \( \mu'(m_r) \), it must be that:

\[ w_\ell \succeq_{\mu'(m_r)} m_r \]

2. Hence:

\[ w_\ell \succeq_{\mu'(m_r)} m_r \]

3. Together with the above, this means \((m_r, w_\ell)\) form a blocking pair for \( \mu' \), a contradiction.

4. Tada!
Proof

\[ m_r \succ_{w_\ell} \mu'(w_\ell) \]

1. Since \( m_r \) had applied to \( w_\ell \) before \( \mu(m_r) \), it must be that:

\[ w_\ell \succ_{m_r} \mu(m_r) \]
Proof

\[ m_r \succ_{w_\ell} \mu'(w_\ell) \]

1. Since \( m_r \) had applied to \( w_\ell \) before \( \mu(m_r) \), it must be that:

\[ w_\ell \succ_{m_r} \mu(m_r) \]

2. Hence:

\[ w_\ell \succ_{m_r} \mu'(m_r) \]
Proof

\[ m_r \succ_{w_\ell} \mu'(w_\ell) \]

1. Since \( m_r \) had applied to \( w_\ell \) before \( \mu(m_r) \), it must be that:

\[ w_\ell \succ_{m_r} \mu(m_r) \]

2. Hence:

\[ w_\ell \succ_{m_r} \mu'(m_r) \]

3. Together with the above, this means \((m_r, w_\ell)\) form a blocking pair for \( \mu' \), a contradiction.
Proof

\[ m_r \succ_w \mu'(w_\ell) \]

1. Since \( m_r \) had applied to \( w_\ell \) before \( \mu(m_r) \), it must be that:

\[ w_\ell \succ_m \mu(m_r) \]

2. Hence:

\[ w_\ell \succ_m \mu'(m_r) \]

3. Together with the above, this means \((m_r, w_\ell)\) form a blocking pair for \( \mu' \), a contradiction.

4. Tada!
Thanks!

See you next class — stay healthy!