# Correlated Equilibria

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- Unlikely... Finding Nash equilibria are as hard as finding general fixed points in the worst case.
- But maybe there is some richer family of equilibria we can shoot for...
- Analogous to our earlier relaxation from *Pure* to *Mixed* equilibria.

# Traffic Lights

Consider the following two player traffic light game that will be familiar to those of you who can drive:

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But one player never gets any utility...

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3. So both players play STOP with probability $p = 100/101$, and play GO with probability $(1-p) = 1/101$.
4. This is even worse! Now both players get payoff 0 in expectation (rather than just one of them), and risk a horrific negative utility.

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The four possible action profiles have roughly the following probabilities under this equilibrium:

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Worse: *there is no set of mixed strategies that creates this distribution over action profiles.*

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5. We can generalize this...

# Correlated Equilibrium

## Definition

A *correlated equilibrium* is a distribution $\mathcal{D}$ over action profiles $A$ such that for every player $i$, every action $\hat{a}_i$, and every action $a_i^*$:

$$\mathrm{E}_{a\sim\mathcal{D}}[u_i(a)|a_i = \hat{a}_i] \geq \mathrm{E}_{a\sim\mathcal{D}}[u_i(a_i^*, a_{-i})|a_i = \hat{a}_i]$$

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For example: Conditioned on seeing STOP, you know your opponent will GO, so STOP is a best response. Conditioned on seeing GO, you know your opponent will STOP, so GO is a best response.

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4. The difference: the suggestion just has to be a best response on average, not *conditioned* on having seen it.
5. Whether it is sensible depends on whether you have to commit to following the correlating device up front, or have the option of deviating after seeing the suggestion.

# Hierarchies

CCE can occasionally suggest obviously bad actions. CE cannot. Consider:

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The payoff for each player for playing according to this distribution is:

$$(1/3) \cdot 1 + (1/3) \cdot 1 - (1/3) \cdot 1.1 = 0.3$$

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the payoff a player would get by playing the fixed action $A$ or $B$ while his opponent randomized would be:

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and the payoff for playing $C$ would be strictly less than zero.

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and the payoff for playing $C$ would be strictly less than zero.

Hence this is a CCE *even though* conditioned on being told to play $C$, it is not a best response. This means that the given distribution is a coarse correlated equilibrium, *but not* a CE.

# Hierarchies

Solution Concept Recap

$$DSE \subset PSNE \subset MSNE \subset CE \subset CCE$$

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1. Starting at MSNE, we have guaranteed existence.
2. Want to show: Starting at CE, we have computational tractability.

# Characterization in Terms of Regret

## Definition

For a *strategy modification rule* $F_i : A_i \to A_i$ and an action profile $a \in A$:

$$\mathrm{Regret}_i(a, F_i) = u_i(F_i(a_i), a_{-i}) - u_i(a)$$

i.e. it is how much player $i$ regrets not applying $F_i$ to change his action.

We say that $F_i$ is a constant strategy modification rule if $F_i(a_i) = F_i(a'_i)$ for all $a_i, a'_i \in A_i$.

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We can give an alternative characterization of CCE:

**Definition**

A distribution $\mathcal{D}$ is a coarse correlated equilibrium if for every player $i$ and for every *constant* strategy modification rule $F_i$:

$$\mathrm{E}_{a \sim \mathcal{D}}[\mathrm{Regret}_i(a, F_i)] \leq 0$$

# Characterization in Terms of Regret

1. An immediate consequence of this definition is that if $a^1, \ldots, a^T$ are a sequence of actions with $\Delta(T)$ regret, then $\bar{a} = \frac{1}{T}\sum_{t=1}^{T} a^t$ forms a $\Delta(T)$-approximate coarse correlated equilibrium.

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2. In particular, if everyone plays an (arbitrary) game with the PW algorithm, after $T$ steps they will have generated a sequence of plays that corresponds to a $\Delta(T) = 2\sqrt{\log k/T}$-approximate CCE
3. Can we approach computing CE in the same way? First step: characterize CE in terms of regret.

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A distribution $\mathcal{D}$ is a correlated equilibrium if for all players $i$ and for *all* strategy modification rules $F_i$:

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To see this:

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3. Are there learning algorithms that efficiently converge to correlated equilibrium?
4. Look for learning algorithms with stronger regret guarantees...

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$$\frac{1}{T}\sum_{t=1}^{T} \ell_{a^t} \leq \frac{1}{T}\sum_{t=1}^{T} \ell_{F(a^t)} + \Delta(T)$$

for all strategy modification rules $F$ and for $\Delta(T) = o(1)$.

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3. "No Swap Regret"
4. We'll see how to do this! (Next lecture).

# Thanks!

See you next class — stay healthy!