Zero Sum Games and the Minimax Thoerem

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- They have a very special property: the minimax theorem.
- And a close connection to the polynomial weights algorithm (and related algorithms)
- Playing the polynomial weights algorithm in a zero sum game leads to equilibrium (a plausible dynamic!)
- In fact, we'll use it to prove the minimax theorem.

Zero Sum Games

Definition

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1. Strictly adversarial games: The only way for player 1 to improve his payoff is to harm player 2, and vice versa.
2. Closely related to linear programming, adversarial machine learning, and lots of other things.

Example

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	Morality	Tax-Cuts
Economy	(3,-3)	(-1,1)
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The row player (Max) wishes to *maximize* the utility. The column player (Min) wishes to *minimize* the utility.

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4. So she plays Tax-cuts.

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$$\arg\max_{p} \min(p \cdot 3 - 2(1-p), p \cdot (-1) + (1-p))$$

5. And if Min goes first, she should play:

$$\arg\min_{q} \max(q \cdot 3 - (1-q), q \cdot (-2) + (1-q))$$

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$$3p - 2(1-p) = -p + (1-p) \Leftrightarrow 5p - 2 = 1 - 2p \Leftrightarrow p = \frac{3}{7}$$

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5. And when Min best responds, he gets payoff $1 - 2p = 1/7$.

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3. i.e. she should play $(q, 1-q)$ such that:

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4. Lets investigate further...

Order of Play

We use the notation $[n] = \{1, 2, \ldots, n\}$, and $\Delta[n]$ to denote the set of probability distributions over $[n]$:

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Definition

For an $n \times m$ matrix U (think about this as the payoff matrix in a two player zero sum game if you like):

$$\max\min(U) = \max_{p \in \Delta[n]} \min_{y \in [m]} \sum_{i=1}^{n} p_i \cdot U(i, y)$$

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If U is a zero sum game, then $\max\min(U)$ represents the payoff that Max can guarantee if he goes first, and $\min\max(U)$ represents the payoff that he can guarantee if Min goes first.

The Minimax Theorem

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In any Nash equilibrium of a zero sum game, Max plays a maxmin strategy and Min plays a minmax strategy. Note that these can be computed without needing to reason about what the other player is doing.

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All Nash equilibria in Zero sum games have the same payoff – the $\max\min$ *value of the game.*

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4. Previously, Borell had proven it for the special case of 5×5 matrices, and thought it was false for larger matrices.
5. But well give an easy, constructive proof.

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2. Write $v_1 = \min\max(U)$ and $v_2 = \max\min(U)$ (And so $v_1 = v_2 + \epsilon$ for some constant $\epsilon > 0$).
3. In other words: if Min has to go first, then Max can guarantee payoff at least v_1, but if Max is forced to go first, then Min can force Max to have payoff only v_2.

Proof: A Thought Experiment

Lets consider what happens when Min and Max repeatedly play against each other as follows, for T rounds:

1. Min will play using the polynomial weights algorithm. i.e. at each round t, the weights w^t of the polynomial weights algorithm will form her mixed strategy, and she will sample an action at random from this distribution, updating based on the losses she experiences at that round.
2. Max will play the best response to Min's strategy. i.e. Max will play $x^t = \arg\max_x \mathrm{E}_{y \sim w^t}[U(x, y)]$.

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What we know about each player's average payoffs when they play in this manner?

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$\bar{x}$ is the mixed strategy that puts weight $1/T$ on each action x^t.
$\Delta(T)$ is the regret bound of the polynomial weights algorithm:

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By definition, $\min_{y^*}\mathrm{E}_{x\sim\bar{x}}U(x,y^*)\le \max\min(U)=v_2$ and so:

$$\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}[U(x^t,y^t)]\le v_2+\Delta(T)$$

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Taking T large enough leads to contradiction.

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3. It does so *without needing to know what the game is*. The game matrix is not an input to the PW algorithm!
4. The only information needed is the realized payoffs are for the actions as it plays the game.

Thanks!

See you next class — stay healthy!