When do Best Response Dynamics Converge?

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- Is that it? How much further can we push it?
- Today: study a couple more games in which BRD converges, and try to abstract what is needed.
- Characterize *exactly* when BRD is guaranteed to converge.

Load Balancing Games on Identical Machines

Definition

A *load balancing game on identical machines* models n players $i \in P$ scheduling jobs of size $w_i > 0$ on m identical machines F. The game has:

1. Action space $A_i = F$ for each player
2. For each machine $j \in F$, a load $\ell_j(a) = \sum_{i:a_i=j} w_i$

The cost of player i is the load of the machine he plays on: $c_i(a) = \ell_{a_i}(a)$.

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Almost a congestion game — but facility costs depend on *which* players are using them.

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Theorem

Best response dynamics converge in load balancing games on identical machines.

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Corollary

Load balancing games on identical machines have pure strategy Nash equilibria

Proof

Define $\phi(a) = \frac{1}{2}\sum_{j=1}^{m} \ell_j(a)^2$. Suppose player i switches from machine j to machine j'. Then we have:

$$\Delta c_i(a) \equiv c_i(j', a_{-i}) - c_i(j, a_{-i})$$

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Note: $\Delta c_i \neq \Delta\phi$.

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1. The players are vertices $P = V$.
2. Each edge $e = (i, j) \in E$ has weight w_e
3. Actions $A_i = \{-1, 1\}$ (read {red, blue})
4. $u_i(a) = \sum_{e=(i,j)\in E} w_e \cdot a_i \cdot a_j = \sum_{j:a_i=a_j} w_{i,j} - \sum_{j:a_i\neq a_j} w_{i,j}$

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"Everyone picks an affiliation, and obtains utility equal to the weight of friends who pick the same affiliation, and disutility equal to the weight of friends who don't."

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The Red-State/Blue-State game always has a pure strategy Nash equilibrium.

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Definition

A function $\phi : A \to \mathbb{R}_{\geq 0}$ is an *exact potential function* for a game G if for all $a \in A$, all i, and all $a_i, b_i \in A_i$:

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Definition

$\phi : A \rightarrow \mathbb{R}_{\geq 0}$ is an *ordinal potential function* for a game G if for all $a \in A$, all i, and all $a_i, b_i \in A_i$:

$$(c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) < 0) \Rightarrow (\phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) < 0)$$

i.e. the change in utility is always equal *in sign* to the change in potential.

A Characterization

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Best response dynamics is guaranteed to converge in a game G if and only if the game has an ordinal potential function.

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So our proof technique is without loss of generality!

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2. Consider a graph $G = (V, E)$:
   2.1 Let each $a \in A$ be a vertex in the graph: i.e. $V = A$.
   2.2 For each pair of vertices $a, b \in V$, add a directed edge (a, b) if it is possible to get to get from b to a via a best response move – i.e. if there is some index i such that $b = (b_i, a_{-i})$, and $c_i(b_i, a_{-i}) < c_i(a)$.

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5. BRD converges = there are no cycles in this graph.

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4. We need: for any edge $a \rightarrow b$, $\phi(b) < \phi(a)$.
5. Its true! $\phi(a) \geq \phi(b) + 1$. (why?)

Thanks!

See you next class — stay healthy!