When do Best Response Dynamics Converge?

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Overview

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- We know best response dynamics (BRD) converges in congestion games.
- Is that it? How much further can we push it?
- Today: study a couple more games in which BRD converges, and try to abstract what is needed.
- Characterize exactly when BRD is guaranteed to converge.
Load Balancing Games on Identical Machines

Definition
A load balancing game on identical machines models $n$ players $i \in P$ scheduling jobs of size $w_i > 0$ on $m$ identical machines $F$. The game has:

1. Action space $A_i = F$ for each player
2. For each machine $j \in F$, a load $\ell_j(a) = \sum_{i : a_i = j} w_i$

The cost of player $i$ is the load of the machine he plays on: $c_i(a) = \ell_{a_i}(a)$. 
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The cost of player $i$ is the load of the machine he plays on: $c_i(a) = \ell_{a_i}(a)$.

*Almost* a congestion game — but facility costs depend on *which* players are using them.
Load Balancing Games on Identical Machines
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Theorem

*Best response dynamics converge in load balancing games on identical machines.*
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Corollary

Load balancing games on identical machines have pure strategy Nash equilibria
Proof

Define $\phi(a) = \frac{1}{2} \sum_{j=1}^{m} \ell_j(a)^2$. Suppose player $i$ switches from machine $j$ to machine $j'$. Then we have:

$$\Delta c_i(a) \equiv c_i(j', a_{-i}) - c_i(j, a_{-i})$$
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Note: $\Delta c_i \neq \Delta \phi$.
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Red State/Blue State Game

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Definition
The Red State/Blue State game is played on a graph $G = (V, E)$.

1. The players are vertices $P = V$.
2. Each edge $e = (i, j) \in E$ has weight $w_e$
3. Actions $A_i = \{-1, 1\}$ (read \{red, blue\})
4. $u_i(a) = \sum_{e=(i,j) \in E} w_e \cdot a_i \cdot a_j = \sum_{j:a_i=a_j} w_{i,j} - \sum_{j:a_i \neq a_j} w_{i,j}$
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“Everyone picks an affiliation, and obtains utility equal to the weight of friends who pick the same affiliation, and disutility equal to the weight of friends who don’t.”
Red State/Blue State Game
Red State/Blue State Game

**Theorem**

*The Red-State/Blue-State game always has a pure strategy Nash equilibrium.*
Proof

Define:

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Abstracting Away...

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**Definition**

A function $\phi : A \rightarrow \mathbb{R}_{\geq 0}$ is an *exact potential function* for a game $G$ if for all $a \in A$, all $i$, and all $a_i, b_i \in A_i$:

$$\phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) = c_i(b_i, a_{-i}) - c_i(a_i, a_{-i})$$
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**Definition**
$\phi : A \rightarrow \mathbb{R}_{\geq 0}$ is an *ordinal potential function* for a game $G$ if for all $a \in A$, all $i$, and all $a_i, b_i \in A_i$:

$$(c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) < 0) \Rightarrow (\phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) < 0)$$

i.e. the change in utility is always equal in *sign* to the change in potential.
A Characterization

Theorem

Best response dynamics is guaranteed to converge in a game $G$ if and only if the game has an ordinal potential function.
A Characterization

Theorem

*Best response dynamics is guaranteed to converge in a game $G$ if and only if the game has an ordinal potential function.*

*So our proof technique is without loss of generality!*
Proof

1. We’ve already seen the forward direction (ordinal potential function \( \Rightarrow \) BRD converges) several times now, so let’s prove the reverse direction.
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2. Consider a graph $G = (V, E)$:
   
   2.1 Let each $a \in A$ be a vertex in the graph: i.e. $V = A$.
   
   2.2 For each pair of vertices $a, b \in V$, add a directed edge $(a, b)$ if it is possible to get to get from $b$ to $a$ via a best response move – i.e. if there is some index $i$ such that $b = (b_i, a_{-i})$, and $c_i(b_i, a_{-i}) < c_i(a)$.

3. BRD can be viewed as traversing this graph: Start at an arbitrary vertex $a$, and then traverse arbitrary outgoing edges.

4. Nash Equilibria are the sinks in this graph.

5. BRD converges = there are no cycles in this graph.
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2. The graph is acyclic, so: from every state $a$ there is some sink $s$ that is reachable. (why?)
3. For each vertex $a$, define $\phi(a)$ to be the length of the longest finite path from $a$ to any sink $s$. 
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4. We need: for any edge $a \rightarrow b$, $\phi(b) < \phi(a)$.
5. Its true! $\phi(a) \geq \phi(b) + 1$. (why?)
Thanks!

See you next class — stay healthy!