Basic Definitions

Aaron Roth

University of Pennsylvania

January 23 2024

Guess 2/3 the average: Winners!

1. Average guess: 15.92

Guess 2/3 the average: Winners!

1. Average guess: 15.92
2. 2/3 the average: 10.61

Guess 2/3 the average: Winners!

1. Average guess: 15.92
2. 2/3 the average: 10.61
3. Closest Guess: 12.1

Guess 2/3 the average: Winners!

1. Average guess: 15.92
2. 2/3 the average: 10.61
3. Closest Guess: 12.1
4. Winner: Cyrus Singer

Guess 2/3 the average stats

1. Guesses above 66: 0

Guess 2/3 the average stats

1. Guesses above 66: 0
2. Guesses above 44: 2

Guess 2/3 the average stats

1. Guesses above 66: 0
2. Guesses above 44: 2
3. Guesses above 29.33: 4

Guess 2/3 the average stats

1. Guesses above 66: 0
2. Guesses above 44: 2
3. Guesses above 29.33: 4
4. Guesses above 19.56: 12

Guess 2/3 the average stats

1. Guesses above 66: 0
2. Guesses above 44: 2
3. Guesses above 29.33: 4
4. Guesses above 19.56: 12
5. Guesses above 13.04: 16

Guess 2/3 the average stats

1. Guesses above 66: 0
2. Guesses above 44: 2
3. Guesses above 29.33: 4
4. Guesses above 19.56: 12
5. Guesses above 13.04: 16
6. ... Guesses of 0: 4

Overview

- Today we'll give (review) the basic definitions that will underly our study this semester.

Overview

- Today we'll give (review) the basic definitions that will underly our study this semester.
- Games, Best Responses, Dominant Strategies, Iterated Elimination...

Overview

- Today we'll give (review) the basic definitions that will underly our study this semester.
- Games, Best Responses, Dominant Strategies, Iterated Elimination...
- Solution concepts: Nash equilibrium

A Game

Definition

A *game* is an interaction defined by:

- A set of players P

A Game

Definition

A *game* is an interaction defined by:

- A set of players P
- A finite set of actions A_i for each player $i \in P$. We write $A = \times_{i=1}^{n} A_i$ to denote the action space for all players, and $A_{-i} = \times_{j \neq i} A_j$ to denote the action space of all players excluding player j.

A Game

Definition

A *game* is an interaction defined by:

- A set of players P
- A finite set of actions A_i for each player $i \in P$. We write $A = \times_{i=1}^{n} A_i$ to denote the action space for all players, and $A_{-i} = \times_{j \neq i} A_j$ to denote the action space of all players excluding player j.
- A utility function $u_i : A \to \mathbb{R}$ for each player $i \in P$.

Utility Maximization

Basic assumption: players will always try and act so as to maximize their utility.

Utility Maximization

Basic assumption: players will always try and act so as to maximize their utility.

Definition

The *best-response* to a set of actions $a_{-i} \in A_{-i}$ for a player i is any action $a_i \in A_i$ that maximizes $u_i(a_i, a_{-i})$:

$$a_i \in \arg\max_{a \in A_i} u_i(a, a_{-i})$$

Interlude

Question: Is game theory just for sociopaths?

Interlude

Question: Is game theory just for sociopaths?
Answer: Not necessarily. (Assumes only that people have consistent preferences)

The General Idea for Prediction

“In any stable situation, all players should be playing a best response.”

The General Idea for Prediction

"In any stable situation, all players should be playing a best response."
(Otherwise, by definition, the situation would not be stable – somebody would want to change their action.)

When are there stable solutions?

Definition

For a player i, an action $a \in A_i$ (weakly) dominates action $a' \in A_i$ if it is always beneficial to play a over a'. That is, if for all $a_{-i} \in A_{-i}$:

$$u_i(a, a_{-i}) \geq u_i(a', a_{-i})$$

and the inequality is strict for some $a_{-i} \in A_{-i}$.

When are there stable solutions?

Definition

For a player i, an action $a \in A_i$ (weakly) dominates action $a' \in A_i$ if it is always beneficial to play a over a'. That is, if for all $a_{-i} \in A_{-i}$:

$$u_i(a, a_{-i}) \geq u_i(a', a_{-i})$$

and the inequality is strict for some $a_{-i} \in A_{-i}$.

Can normally eliminate dominated strategies from consideration – there is never a situation in which they are the (unique) best response.

Dominant Strategies

Definition

An action $a \in A_i$ is *dominant* for player i if it weakly dominates all actions $a' \neq a \in A_i$.

Dominant Strategies

Definition

An action $a \in A_i$ is *dominant* for player i if it weakly dominates all actions $a' \neq a \in A_i$.

1. A very strong guarantee – Always a best response.

Dominant Strategies

Definition

An action $a \in A_i$ is *dominant* for player i if it weakly dominates all actions $a' \neq a \in A_i$.

1. A very strong guarantee – Always a best response.
2. No need to reason about what your opponents are doing.

Dominant Strategy Equilibrium

Dominant strategies normally don't exist, but when they do, predictions are easy.

Dominant Strategy Equilibrium

Dominant strategies normally don't exist, but when they do, predictions are easy.

Definition

An action profile $a = (a_1, \ldots, a_n) \in A$ is a *dominant strategy equilibrium* of the game $(P, \{A_i\}, \{u_i\})$ if for every $i \in P$, a_i is a dominant strategy for player i.

Example: Prisoner's Dilemma

	Confess	Silent
Confess	$(1, 1)$	$(5, 0)$
Silent	$(0, 5)$	$(3, 3)$

Figure: Prisoner's Dilemma

Example: Prisoner's Dilemma

	Confess	Silent
Confess	$(1, 1)$	$(5, 0)$
Silent	$(0, 5)$	$(3, 3)$

Figure: Prisoner's Dilemma

(Confess, Confess) is a dominant strategy equilibrium is Prisoner's Dilemma.

What if there are no dominant strategies?

- It still makes sense to eliminate *dominated* strategies from consideration.

What if there are no dominant strategies?

- It still makes sense to eliminate *dominated* strategies from consideration.
- Sometimes, once you've done this, new strategies have become dominated.

What if there are no dominant strategies?

- It still makes sense to eliminate *dominated* strategies from consideration.
- Sometimes, once you've done this, new strategies have become dominated.
- We can consider eliminating dominated strategies *iteratively*.

What if there are no dominant strategies?

- It still makes sense to eliminate *dominated* strategies from consideration.
- Sometimes, once you've done this, new strategies have become dominated.
- We can consider eliminating dominated strategies *iteratively*.
- If we are lucky, "iterated elimination of dominated strategies" leads to a unique surviving strategy profile.

Iterated Elimination: Example 1

	X	Y
A	$(5, 2)$	$(4, 2)$
B	$(3, 1)$	$(3, 2)$
C	$(2, 1)$	$(4, 1)$
D	$(4, 3)$	$(5, 4)$

Figure: Example 1

Iterated Elimination: Example 2

	V	W	X	Y	Z
A	$(4, -1)$	$(3, 0)$	$(-3, 1)$	$(-1, 4)$	$(-2, 0)$
B	$(-1, 1)$	$(2, 2)$	$(2, 3)$	$(-1, 0)$	$(2, 5)$
C	$(2, 1)$	$(-1, -1)$	$(0, 4)$	$(4, -1)$	$(0, 2)$
D	$(1, 6)$	$(-3, 0)$	$(-1, 4)$	$(1, 1)$	$(-1, 4)$
E	$(0, 0)$	$(1, 4)$	$(-3, 1)$	$(-2, 3)$	$(-1, -1)$

Figure: Example 2

What if Iterated Elimination Doesn't Eliminate Anything?

We can still ask for a "stable" profile of actions.

What if Iterated Elimination Doesn't Eliminate Anything?

We can still ask for a "stable" profile of actions.

Definition

A profile of actions $a = (a_1, \ldots, a_n) \in A$ is a *pure strategy Nash Equilibrium* if for each player $i \in P$ and for all $a'_i \in A_i$:

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$$

i.e. simultaneously, all players are playing a best response to one another.

What if Iterated Elimination Doesn't Eliminate Anything?

We can still ask for a "stable" profile of actions.

Definition

A profile of actions $a = (a_1, \ldots, a_n) \in A$ is a *pure strategy Nash Equilibrium* if for each player $i \in P$ and for all $a'_i \in A_i$:

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$$

i.e. simultaneously, all players are playing a best response to one another.

Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

What if Iterated Elimination Doesn't Eliminate Anything?

We can still ask for a "stable" profile of actions.

Definition

A profile of actions $a = (a_1, \ldots, a_n) \in A$ is a *pure strategy Nash Equilibrium* if for each player $i \in P$ and for all $a'_i \in A_i$:

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$$

i.e. simultaneously, all players are playing a best response to one another.

Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

Proof.

Homework! □

Problem 1: They don't always exist.

	Heads	Tails
Heads	$(1, -1)$	$(-1, 1)$
Tails	$(-1, 1)$	$(1, -1)$

Figure: Matching Pennies

Problem 2: They aren't always unique.

	Bach	Stravinsky
Bach	$(5, 1)$	$(0, 0)$
Stravinsky	$(0, 0)$	$(1, 5)$

Figure: Bach of Stravinsky

Question: What to Predict when No Pure Nash Equilibria?

Definition

A two-player game is *zero-sum* if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of of both players sum to zero at every action profile)

Question: What to Predict when No Pure Nash Equilibria?

Definition

A two-player game is *zero-sum* if for all $a \in A$, $u_1(a) = -u_2(a)$.
(i.e. the utilities of of both players sum to zero at every action profile)

1. e.g. Matching Pennies.

Question: What to Predict when No Pure Nash Equilibria?

Definition

A two-player game is *zero-sum* if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of of both players sum to zero at every action profile)

1. e.g. Matching Pennies.
2. In matching pennies you should randomize to thwart your opponent: Flip a coin and play heads 50% of the time, and tails 50% of the time.

Question: What to Predict when No Pure Nash Equilibria?

Definition

A two-player game is *zero-sum* if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of of both players sum to zero at every action profile)

1. e.g. Matching Pennies.
2. In matching pennies you should randomize to thwart your opponent: Flip a coin and play heads 50% of the time, and tails 50% of the time.

Definition

A *mixed-strategy* $p_i \in \Delta A_i$ is a probability distribution over actions $a_i \in A_i$: i.e. a set of numbers $p_i(a_i)$ such that:

1. $p_i(a_i) \geq 0$ for all $a_i \in A_i$
2. $\sum_{a_i \in A_i} p_i(a_i) = 1$.

For $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$, we write:

$$u_i(p) = E_{a_i \sim p_i}[u_i(a)]$$

Mixed Strategy Nash Equilibria

Definition

A *mixed strategy Nash equilibrium* is a tuple $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$ such that for all i, and for all $a_i \in A_i$:

$$u_i(p_1, p_{-i}) \geq u_i(a_i, p_{-i})$$

Mixed Strategy Nash Equilibria

Definition

A *mixed strategy Nash equilibrium* is a tuple $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$ such that for all i, and for all $a_i \in A_i$:

$$u_i(p_1, p_{-i}) \geq u_i(a_i, p_{-i})$$

Theorem (Nash)

Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.

Mixed Strategy Nash Equilibria

Definition

A *mixed strategy Nash equilibrium* is a tuple $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$ such that for all i, and for all $a_i \in A_i$:

$$u_i(p_1, p_{-i}) \geq u_i(a_i, p_{-i})$$

Theorem (Nash)

Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.

But... The proof is non-constructive, so its not necessarily clear how to find one of these, even though they exist

Thanks!

See you next class — stay healthy!