Basic Definitions

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University of Pennsylvania

January 23 2024
Guess 2/3 the average: Winners!

1. Average guess: 15.92
Guess 2/3 the average: Winners!

1. Average guess: 15.92
2. 2/3 the average: 10.61

Winner: Cyrus Singer
Guess 2/3 the average: Winners!

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3. Closest Guess: 12.1

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1. Average guess: 15.92
2. 2/3 the average: 10.61
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4. Winner: Cyrus Singer
Guess 2/3 the average stats

1. Guesses above 66: 0
Guess 2/3 the average stats

1. Guesses above 66: 0
2. Guesses above 44: 2
Guess 2/3 the average stats

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3. Guesses above 29.33: 4
Guess 2/3 the average stats

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4. Guesses above 19.56: 12
Guess 2/3 the average stats

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3. Guesses above 29.33: 4
4. Guesses above 19.56: 12
5. Guesses above 13.04: 16
Guess 2/3 the average stats

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3. Guesses above 29.33: 4
4. Guesses above 19.56: 12
5. Guesses above 13.04: 16
6. ... Guesses of 0: 4
Overview

Today we’ll give (review) the basic definitions that will underly our study this semester.
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- Games, Best Responses, Dominant Strategies, Iterated Elimination...
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- Games, Best Responses, Dominant Strategies, Iterated Elimination...
- Solution concepts: Nash equilibrium
A Game

Definition
A game is an interaction defined by:

- A set of players $P$

- A finite set of actions $A_i$ for each player $i \in P$. We write $A = \times_{i=1}^n A_i$ to denote the action space for all players, and $A - j = \times_{j \neq i} A_j$ to denote the action space of all players excluding player $j$.

- A utility function $u_i : A \rightarrow \mathbb{R}$ for each player $i \in P$. 
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▶ A finite set of actions $A_i$ for each player $i \in P$. We write $A = \times_{i=1}^{n} A_i$ to denote the action space for all players, and $A_{-i} = \times_{j \neq i} A_j$ to denote the action space of all players excluding player $j$. 
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▶ A utility function \( u_i : A \to \mathbb{R} \) for each player \( i \in P \).
Utility Maximization

Basic assumption: players will always try and act so as to maximize their utility.
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Definition
The *best-response* to a set of actions $a_{-i} \in A_{-i}$ for a player $i$ is any action $a_i \in A_i$ that maximizes $u_i(a_i, a_{-i})$:

$$a_i \in \arg \max_{a \in A_i} u_i(a, a_{-i})$$
Interlude

**Question:** Is game theory just for sociopaths?
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**Answer:** Not necessarily. (Assumes only that people have consistent preferences)
The General Idea for Prediction

“In any stable situation, all players should be playing a best response.”
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“In any stable situation, all players should be playing a best response.”
(Otherwise, by definition, the situation would not be stable – somebody would want to change their action.)
When are there stable solutions?

Definition
For a player $i$, an action $a \in A_i$ (weakly) dominates action $a' \in A_i$ if it is always beneficial to play $a$ over $a'$. That is, if for all $a_{-i} \in A_{-i}$:

$$u_i(a, a_{-i}) \geq u_i(a', a_{-i})$$

and the inequality is strict for some $a_{-i} \in A_{-i}$. 

Can normally eliminate dominated strategies from consideration – there is never a situation in which they are the (unique) best response.
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Definition
An action \(a \in A_i\) is *dominant* for player \(i\) if it weakly dominates all actions \(a' \neq a \in A_i\).
Dominant Strategies

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1. A very strong guarantee – Always a best response.
Dominant Strategies

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An action $a \in A_i$ is dominant for player $i$ if it weakly dominates all actions $a' \neq a \in A_i$.

1. A very strong guarantee – Always a best response.
2. No need to reason about what your opponents are doing.
Dominant Strategy Equilibrium

Dominant strategies normally don’t exist, but when they do, predictions are easy.
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Dominant strategies normally don’t exist, but when they do, predictions are easy.

Definition
An action profile $a = (a_1, \ldots, a_n) \in A$ is a dominant strategy equilibrium of the game $(P, \{A_i\}, \{u_i\})$ if for every $i \in P$, $a_i$ is a dominant strategy for player $i$. 
Example: Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Silent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>(1, 1)</td>
<td>(5, 0)</td>
</tr>
<tr>
<td>Silent</td>
<td>(0, 5)</td>
<td>(3, 3)</td>
</tr>
</tbody>
</table>

Figure: Prisoner’s Dilemma
Example: Prisoner’s Dilemma

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Figure: Prisoner’s Dilemma

(Confess, Confess) is a dominant strategy equilibrium is Prisoner’s Dilemma.
What if there are no dominant strategies?

▶ It still makes sense to eliminate *dominated* strategies from consideration.
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- We can consider eliminating dominated strategies *iteratively.*
What if there are no dominant strategies?

▶ It still makes sense to eliminate \textit{dominated} strategies from consideration.

▶ Sometimes, once you’ve done this, new strategies have become dominated.

▶ We can consider eliminating dominated strategies \textit{iteratively}.

▶ If we are lucky, “iterated elimination of dominated strategies” leads to a unique surviving strategy profile.
### Iterated Elimination: Example 1

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(5, 2)</td>
<td>(4, 2)</td>
</tr>
<tr>
<td>B</td>
<td>(3, 1)</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>C</td>
<td>(2, 1)</td>
<td>(4, 1)</td>
</tr>
<tr>
<td>D</td>
<td>(4, 3)</td>
<td>(5, 4)</td>
</tr>
</tbody>
</table>

**Figure:** Example 1
## Iterated Elimination: Example 2

<table>
<thead>
<tr>
<th></th>
<th>V</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(4, −1)</td>
<td>(3, 0)</td>
<td>(−3, 1)</td>
<td>(−1, 4)</td>
<td>(−2, 0)</td>
</tr>
<tr>
<td>B</td>
<td>(−1, 1)</td>
<td>(2, 2)</td>
<td>(2, 3)</td>
<td>(−1, 0)</td>
<td>(2, 5)</td>
</tr>
<tr>
<td>C</td>
<td>(2, 1)</td>
<td>(−1, −1)</td>
<td>(0, 4)</td>
<td>(4, −1)</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>D</td>
<td>(1, 6)</td>
<td>(−3, 0)</td>
<td>(−1, 4)</td>
<td>(1, 1)</td>
<td>(−1, 4)</td>
</tr>
<tr>
<td>E</td>
<td>(0, 0)</td>
<td>(1, 4)</td>
<td>(−3, 1)</td>
<td>(−2, 3)</td>
<td>(−1, −1)</td>
</tr>
</tbody>
</table>

**Figure:** Example 2
What if Iterated Elimination Doesn’t Eliminate Anything?

We can still ask for a “stable” profile of actions.
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Definition

A profile of actions \( a = (a_1, \ldots, a_n) \in A \) is a pure strategy Nash Equilibrium if for each player \( i \in P \) and for all \( a'_i \in A_i \):

\[
u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})\]

i.e. simultaneously, all players are playing a best response to one another.

Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

Proof. Homework!
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Homework!
Problem 1: They don’t always exist.

<table>
<thead>
<tr>
<th>Heads</th>
<th>Heads (1, −1)</th>
<th>Tails (−1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tails</td>
<td>(−1, 1)</td>
<td>(1, −1)</td>
</tr>
</tbody>
</table>

Figure: Matching Pennies
Problem 2: They aren’t always unique.

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bach</td>
<td>(5, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>(0, 0)</td>
<td>(1, 5)</td>
</tr>
</tbody>
</table>

**Figure:** Bach of Stravinsky
Question: What to Predict when No Pure Nash Equilibria?

Definition
A two-player game is zero-sum if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of both players sum to zero at every action profile)
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2. In matching pennies you should randomize to thwart your opponent: Flip a coin and play heads 50% of the time, and tails 50% of the time.
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Definition
A mixed-strategy \( p_i \in \Delta A_i \) is a probability distribution over actions \( a_i \in A_i \): i.e. a set of numbers \( p_i(a_i) \) such that:

1. \( p_i(a_i) \geq 0 \) for all \( a_i \in A_i \)
2. \( \sum_{a_i \in A_i} p_i(a_i) = 1 \).

For \( p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n \), we write:

\[
u_i(p) = E_{a_i \sim p_i}[u_i(a)]\]
Mixed Strategy Nash Equilibria

Definition

A *mixed strategy Nash equilibrium* is a tuple

\[ p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n \]

such that for all \( i \), and for all \( a_i \in A_i \):

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Theorem (Nash)
Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.
Mixed Strategy Nash Equilibria

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A mixed strategy Nash equilibrium is a tuple
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But... The proof is non-constructive, so its not necessarily clear how to find one of these, even though they exist
Thanks!

See you next class — stay healthy!