Mechanism Design via Differential Privacy

Aaron Roth

University of Pennsylvania

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Overview

- We’ll have one final lecture using digital goods auctions as a testbed for techniques in mechanism design.

Recall two lectures ago: we designed a dominant strategy truthful auction that for any vector of valuations $v$, obtained revenue:

$$\text{Rev}(v) \geq \text{OPT}(v) - O(\sqrt{n})$$

where $\text{OPT}(v) = \max_{p \in [0, 1]} p \cdot |\{i : v_i \geq p\}|$.

This class: we will relax our solution concept to asymptotic dominant strategy truthfulness, and try to obtain a better revenue guarantee.

Our tool: Differential privacy, a technique developed for protecting user privacy in data analysis.
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- This class: we will relax our solution concept to asymptotic dominant strategy truthfulness, and try to obtain a better revenue guarantee.
- Our tool: Differential privacy, a technique developed for protecting user privacy in data analysis.
Approach

- Recall: Why can we not simply compute the price $p^* = \arg \max_{p \in [0,1]} p \cdot |\{i : v_i \geq p\}$ and charge that?

- This price will be highly manipulable by (at least) one of the bidders – $p^* = v_{i^*}$ for some bidder $i^*$, who will have strong incentive to change his bid.

- So to get dominant strategy truthfulness, we needed to compute prices that were independent of bidder reports.

- But what if we could compute a price $p$ that is almost independent of the reported valuation $v_i$ for every buyer $i$?

- Will this yield in some sense an approximate truthfulness guarantee? This will be the idea behind our approach.
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Privacy Definitions

Definition
Two bid vectors $v, v' \in [0, 1]^n$ are neighbors if they differ in just a single agent’s bid: i.e. if there exists an index $i$ such that $v_j = v'_j$ for every index $j \neq i$. 

We can now define differential privacy:

Definition
A mechanism $M : [0, 1]^n \rightarrow O$ is $\epsilon$-differentially private if for every pair of neighboring bid vectors $v, v' \in [0, 1]^n$, and for every outcome $x \in O$:

$$\Pr[M(v) = x] \leq \exp(\epsilon) \Pr[M(v') = x]$$

Here you should think of $\epsilon < 1$ as a small constant, and think of $\exp(\epsilon) \approx (1 + \epsilon)$. For $\epsilon \leq 1$ we have:

$$1 + \epsilon \leq \exp(\epsilon) \leq 1 + 2\epsilon$$
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Approximate Truthfulness

We can also define what we mean by *approximate* dominant strategy truthfulness:

\[ \text{A mechanism } M : \{0, 1\}^n \rightarrow O \text{ is } \epsilon \text{-approximately dominant strategy truthful if for every bidder } i, \text{ every utility function } u_i : \{0, 1\} \times O \rightarrow \{0, 1\}, \text{ every vector of valuations } v \in \{0, 1\}^n, \text{ and every deviation } v'_i \in \{0, 1\}, \text{ if we write } v'_i = (v - i, v'_i), \text{ then:} \]

\[ E_{o \sim M}(u_i(v_i, o)) \geq E_{o \sim M}(u_i(v'_i, o)) - \epsilon \]

In other words, we require that no bidder can substantially (by more than \( \epsilon \)) increase his utility by mis-reporting his valuation.
Approximate Truthfulness

We can also define what we mean by *approximate* dominant strategy truthfulness:

**Definition**
A mechanism $M : [0, 1]^n \rightarrow O$ is $\epsilon$-approximately dominant strategy truthful if for every bidder $i$, every utility function $u_i : [0, 1] \times O \rightarrow [0, 1]$, every vector of valuations $v \in [0, 1]^n$, and every deviation $v'_i \in [0, 1]$, if we write $v' = (v_{-i}, v'_i)$, then:

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E_{o \sim M(v)}[u_i(v_i, o)] \geq E_{o \sim M(v')}[u_i(v_i, o)] - \epsilon
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**Theorem**

*If a mechanism $M : [0, 1]^n \rightarrow O$ is $\epsilon$-differentially private, then $M$ is also $\epsilon$-approximately dominant strategy truthful.*
The Connection: Proof

Fix any buyer $i$, valuation vector $v$, and utility function $u_i : [0, 1] \times \mathcal{O} \rightarrow [0, 1]$. 

\[ E_{o \sim M(v)} [u_i(v, o)] = \sum_{o \in \mathcal{O}} u_i(v, o) \cdot \Pr[M(v) = o] \geq \sum_{o \in \mathcal{O}} u_i(v, o) \cdot \exp(-\epsilon) \Pr[M(v') = o] = \exp(-\epsilon) E_{o \sim M(v')} [u_i(v, o)] \geq E_{o \sim M(v')} [u_i(v, o)] - \epsilon \]

The last inequality follows because for $\epsilon < 1$, $\exp(-\epsilon) \geq 1 - \epsilon$, and $u_i(v, o) \leq 1$. 

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Fix any buyer $i$, valuation vector $v$, and utility function $u_i : \mathcal{O} \rightarrow [0, 1]$.

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$$\geq \sum_{o \in \mathcal{O}} u_i(v_i, o) \cdot \exp(-\epsilon) \Pr[M(v') = o]$$
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The last inequality follows because for \( \epsilon < 1 \), \( \exp(-\epsilon) \geq 1 - \epsilon \), and \( u_i(v_i, o) \leq 1 \).
So: to design an approximately truthful mechanism that guarantees high revenue, it is sufficient to design a differentially private mechanism with high revenue.
Exploiting the Connection

- So: to design an approximately truthful mechanism that guarantees high revenue, it is sufficient to design a differentially private mechanism with high revenue.
- Let's try a straightforward approach: directly picking a price that approximately maximizes revenue for the reported bidder valuations.
So: to design an approximately truthful mechanism that guarantees high revenue, it is sufficient to design a differentially private mechanism with high revenue.

Let's try a straightforward approach: directly picking a price that approximately maximizes revenue for the reported bidder valuations.

As in the last two lectures, let's pick a finite subset of prices $P \subset [0, 1]$ to select from.
Consider the following mechanism (an instantiation of what is called “the exponential mechanism” in its more general form):
Consider the following mechanism (an instantiation of what is called “the exponential mechanism” in its more general form): \textbf{ExpMech}(v, \epsilon, P):

\textbf{Define} \ Rev(p, v) = p \cdot |\{i : v_i \geq p\}|.

\textbf{Output} each \( p \in P \) according to the following probability distribution:

\[
Pr[p] = \frac{1}{\phi(v)} \exp \left( \frac{\epsilon \cdot Rev(p, v)}{2} \right)
\]

where

\[
\phi(v) = \sum_{p \in P} \exp \left( \frac{\epsilon \cdot Rev(p, v)}{2} \right)
\]
Theorem

For any $\epsilon, P$: $\text{ExpMech}(\cdot, \epsilon, P)$ is $\epsilon$-differentially private.
Privacy/Truthfulness

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For any $\epsilon, P$: $\text{ExpMech}(\cdot, \epsilon, P)$ is $\epsilon$-differentially private.
(and thus $\epsilon$-approximately dominant strategy truthful)
Proof

Fix any pair of neighboring bid vectors $v, v'$ and any output $p$. We have:
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$$\Pr[EM(v, \epsilon, P) = p] = \frac{1}{\phi(v)} \exp \left( \frac{\epsilon \cdot \text{Rev}(p, v)}{2} \right)$$
Proof

Fix any pair of neighboring bid vectors $v, v'$ and any output $p$. We have:

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$$\leq \frac{1}{\phi(v')} \exp\left(\frac{\epsilon \cdot (Rev(p, v') + 1)}{2}\right)$$
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$$= \frac{1}{\phi(v)} \exp \left( \frac{\epsilon}{2} \right) \exp \left( \frac{\epsilon \cdot Rev(p, v')}{2} \right)$$
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$$= \exp(\epsilon) \Pr[EM(v', \epsilon, P) = p]$$
Theorem

For any \( P, v, \epsilon, \delta \), with probability \( 1 - \delta \), \( \text{ExpMech}(v, \epsilon, P) \) outputs a price \( p \) such that:

\[
\begin{align*}
\text{Rev}(p, v) & \geq \max_{p^* \in P} \text{Rev}(p^*, v) - \frac{2}{\epsilon} \cdot \ln \left( \frac{|P|}{\delta} \right)
\end{align*}
\]
Proof

Let $p^* = \max_{p^* \in P} \text{Rev}(p^*, \nu)$. For any value $x$, we have:

$\Pr_p[\text{Rev}(p, \nu) \leq x] \leq \Pr_p[\text{Rev}(p^*, \nu) = \text{Rev}(p^*, \nu)] \leq |P| \cdot \exp(-\frac{\epsilon \cdot \text{Rev}(p^*, \nu)}{2})$

Now choose $x = \text{Rev}(p^*, \nu) - 2 \epsilon \cdot \ln |P| / \delta$. Plugging that in above, we get:

$\Pr_p[\text{Rev}(p, \nu) \leq x] \leq |P| \cdot \exp(-\frac{\ln |P|}{2}) \delta = \delta$
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Let $p^* = \max_{p^* \in P} \operatorname{Rev}(p^*, v)$. For any value $x$, we have:

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\leq \frac{|P| \cdot \exp(\varepsilon x/2)}{\exp(\varepsilon \text{Rev}(p^*, \nu)/2)}
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Putting it all Together

- We have: an approximately truthful way to select a revenue maximizing price from a finite set of prices $P$, with revenue guarantees with respect to the best price in $P$ that degrade with $|P|$. 

- This is a familiar tradeoff.

- Now our dependence on $|P|$ is only logarithmic...

- Let's again see what happens when we take the natural discretization: $P = \{\alpha, 2\alpha, 3\alpha, \ldots, 1\}$

- Just as before, $|P| = 1/\alpha$, and we have the guarantee that for all $v$: $\max_{p \in P} Rev(p, v) \geq \max_{p \in [0, 1]} Rev(p, v) - \alpha n$. 
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$$\max_{p \in P} \text{Rev}(p, v) \geq \max_{p \in [0,1]} \text{Rev}(p, v) - \alpha n$$
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- Combining our bounds we see that if we discretize the price space by $\alpha$, with probability 0.99, we obtain revenue:

$$\operatorname{Rev}(p, v) \geq \operatorname{OPT} - \alpha \cdot n - O\left(\frac{1}{\epsilon} \ln \left(\frac{1}{\alpha}\right)\right)$$
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$$\text{Rev}(p, v) \geq \text{OPT} - \alpha \cdot n - O\left(\frac{1}{\epsilon} \ln \left(\frac{1}{\alpha}\right)\right)$$

- Choosing $\alpha = 1/n$, we find that for any $\epsilon$, we can obtain an $\epsilon$-approximately dominant strategy truthful mechanism which obtains revenue:

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- Choosing $\alpha = 1/n$, we find that for any $\epsilon$, we can obtain an $\epsilon$-approximately dominant strategy truthful mechanism which obtains revenue:

$$\text{Rev}(p, v) \geq \text{OPT} - O\left(\frac{\log n}{\epsilon}\right)$$

- If we take e.g. $\epsilon = O(1/\log(n))$, then we have an asymptotically truthful mechanism (in the sense that it becomes exactly truthful as $n \to \infty$) that improves by an exponential factor on the revenue guarantee that we were able to obtain with an exactly truthful mechanism for the same problem.
Thanks!

See you next class — stay healthy!