Auction Design

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Overview

Last lecture, we studied *pricing equilibria*. In this lecture, we continue our study of money as a means of exchange, from the perspective of mechanism design. Specifically, we begin our study of how to design *auctions*, which will be mechanisms for choosing outcomes, while managing the incentives of individuals to report to the mechanism their true preferences.
We will consider a very general setting:

- We have a set of possible *alternatives* $A$ that we want to choose from.

- We have a set of $n$ agents $i$ each of whom have a valuation function $v_i \in V$. Each valuation function $v_i: A \to \mathbb{R} \geq 0$.

- An outcome $o = (a, p)$ denotes an alternative $a \in A$ together with a payment vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ specifying a payment $p_i$ for each agent.

- Agents have quasilinear utility functions. The utility that agent $i$ experiences for outcome $o = (a, p)$ is:
  $$u_i(o) = v_i(a) - p_i$$
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This could represent many things. e.g.

- A single item allocation problem. \( a \) represents who gets the good.
- A multi-item allocation problem. \( a \) represents a mapping from people to goods.
- A public goods problem. \( a \) represents whether or not a library is built.
- ...
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**Definition**

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2. A *payment rule* $P : V^n \rightarrow \mathbb{R}^n$
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Let’s lay out a “wish list” of desiderata that our dream auction would satisfy:
Desideratum 1: Safety

Definition (Individual Rationality)
A mechanism is individually rational (IR) if for every agent $i$ and for every $\nu \in V^n$:

$$v_i(X(\nu)) \geq P(\nu)_i$$

i.e. nobody is ever asked to pay more than their (reported) value for the outcome.
Desideratum 2: Incentive Compatibility

Definition (Dominant Strategy Truthfulness)

A mechanism is dominant strategy truthful if for every agent $i$, for every $v \in V^n$, and for every alternative report $v'_i \in V$, we have:

$$u_i(X(v), P(v)) \geq u_i(X(v'_i, v_{-i}), P(v'_i, v_{-i}))$$

or equivalently:

$$v_i(X(v)) - P(v)_i \geq v_i(X(v'_i, v_{-i})) - P(v'_i, v_{-i})_i$$
Desideratum 3: Outcome Quality

Definition (Allocative Efficiency)

A mechanism is *allocatively efficient*, or “Social Welfare Maximizing”, if for all \( v \in V^n \), if \( a = X(v) \), then for all \( a' \in A \) we have:

\[
\sum_i v_i(a) \geq \sum_i v_i(a')
\]
Desideratum 4: Budget Balance

Definition (No Deficit)

A mechanism is *no deficit* if for all \( v \in V^n \):

\[
\sum_{i} P(v)_i \geq 0
\]

i.e. in total, the mechanism does not have to pay to run the auction.
Example: Single Item Auction

1. \( A = [n] \) (representing which of the \( n \) agents get the single item for sale).

2. Valuations are single dimensional. Abusing notation: \( V = \mathbb{R} \geq 0 \), which we will take to mean: 
   \[ v_i(a) = v_i, \quad a = i; \ 0, \quad \text{otherwise}. \]

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For allocative efficiency: must choose $X(v) = \arg\max_i v_i$. What about the payment rule?

1. By individual rationality, we must have $p(v)_j \leq 0$ for all $j \neq X(v)$. Let's try $p(v)_j = 0$, so it only remains to fix $p(v)_i$ for $i = X(v)$. Similarly, we know $p(v)_i \leq v_i$.

2. We could try $p(v)_i = v_i$. Does this lead to an incentive compatible auction?

3. What about $p(v)_i = \arg\max_{j \neq X(v)} v_j$. Is this incentive compatible?
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- Note that its the same thing as the TV “English Auction”.
- What about other pricing rules? What if the winner pays the 3rd highest price?
- Lets see if we can generalize this beyond single item auctions...
The Groves Mechanism

Definition
The *Groves Mechanism* has choice rule:

\[ X(v) = \arg \max_{a \in A} \sum_i v_i(a) \]

and payment rule:

\[ P(v)_i = h_i(v_{-i}) - \sum_{j \neq i} v_j(a^*) \]

where \( h_i \) is an arbitrary function (crucially, independent of \( v_i \)), and \( a^* = X(v) \) is the socially optimal outcome.

Note that the Groves mechanism is a family of mechanisms, instantiated by a choice of \( h_i \).
Two Desiderata

Theorem
*The Groves mechanism is dominant strategy incentive compatible and Allocatively efficient.*

Proof.
It is allocatively efficient by definition, so it remains to verify that it is dominant strategy incentive compatible.
Two Desiderata

Proof.
Fix any agent $i$, and reports $v_{-i}$ of the other players. We have:

$$u_i(X(v), P(v)) = v_i(a^*) + \sum_{j \neq i} v_j(a^*) - h_i(v_{-i})$$

where $a^* = \arg \max_{a \in A} \left( \sum_{j \neq i} v_i(a) + v'_i(a) \right)$. Agent $i$ wishes to report $v'_i$ to maximize his utility.
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Note that $h_i(v_{-i})$ has no dependence on his report, so equivalently, agent $i$ wishes to report $v'_i$ to maximize:

$$v_i(a^*) + \sum_{j \neq i} v_j(a^*) = \sum_{i} v_i(a^*)$$

But note that if agent $i$ truthfully reports $v'_i = v_i$, then $a^*$ maximizes this quantity by definition. Hence, it is a dominant strategy for all agents to report truthfully.
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Fix any agent $i$, and reports $v_{\_i}$ of the other players. We have:

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Intuition

The payment scheme aligns the incentives of the agents and the mechanism designer: both prefer higher social welfare outcomes.
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- Take $h_i(v_{-i}) = 0$ for all $i$. Suppose we have two bidders, with values for the item $v_1 = 5$ and $v_2 = 8$. Truthful bidding results in $X(v) = 2$, resulting in social welfare 8. The payment rule mandates:

$$P(v_1) = -8 \quad P(v_2) = 0$$

Both bidders get utility 8 and have no beneficial deviations. Individual rationality! But the auction is not no-deficit: pays the losing bidder $8.

- How can we pick $h$ to achieve the no-deficit property without breaking individual rationality?
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- Consider a single item auction \((A = [n])\).
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The VCG Mechanism

Definition (The Vickrey-Clarke-Groves (VCG) Mechanism)

The VCG mechanism is an instantiation of the Groves mechanism with

\[ h_i(v_{-i}) = \sum_{j \neq i} v_j(a^*_{-i}) \]

where \( a^*_{-i} = \arg \max_{a \in A} \sum_{j \neq i} v_j(a) \) is the alternative that maximizes social welfare among all agents other than agent \( i \). In other words, the VCG mechanism has payment rule:

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We will show that the VCG mechanism satisfies all of our desiderata.
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Theorem
The VCG mechanism is allocatively efficient and dominant strategy incentive compatible.

Proof.
It is an instantiation of the Groves mechanism.
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The VCG mechanism is individually rational.

Proof.
We need to show that Agent $i$’s utility satisfies:

$$u_i(o) = v_i(a^*) + \sum_{j \neq i} v_i(a^*) - \sum_{j \neq i} v_i(a^*_{-i}) \geq 0$$
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But note that if this is not the case, since $v_i$ is non-negative, we would have:

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But this would contradict the allocative efficiency of \( a^* \)!
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We will in fact show the stronger claim that for all $i$, $P(v)_i \geq 0$. Recall that:

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But note that this is always the case, since \( a^*_i \) is explicitly defined to be the maximizer of \( \sum_{j \neq i} v_j(a) \) over all \( a \in A \).
Wrapping Up

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- So the VCG mechanism satisfies all of our wildest dreams, in an extremely general setting! Can end the class here?
- Not quite – we will see that the VCG mechanism still leaves a bit to be desired. It doesn’t maximize other objectives (like e.g. revenue), and it isn’t always computationally efficient.
Thanks!

See you next class — stay healthy!