Walrasian Equilibrium

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- An “exchange” and a “matching” problem.
- This lecture: We’ll bring money into the picture in a matching like problem.
- And give a formalization of Adam Smith’s “Invisible Hand”
- The thesis (in our simple model): simple, decentralized market dynamics lead to efficient outcomes.
A Model

Suppose we have:

1. $m$ goods $G$ for sale
2. $n$ buyers $i$ who each have valuation functions over bundles, $v_i: 2^G \to [0, 1]$.

Buyers have quasi-linear utility functions: If each good $j \in G$ has a price $p_j$, then a buyer $i$ gets utility for buying a bundle $S \subseteq G$:

$$u_i(S) = v_i(S) - \sum_{j \in S} p_j$$

Questions:
How we should price and allocate goods so that everyone is happy with their allocation. Is this even possible? If it is, can we do so and also achieve a high welfare allocation?
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Some Definitions

First, feasibility:

**Definition**

An allocation $S_1, \ldots, S_n \subseteq G$ is *feasible* if for all $i \neq j$, $S_i \cap S_j = \emptyset$

We write $\text{OPT}$ to denote the socially optimal feasible allocation:

$$\text{OPT} = \max_{S_1, \ldots, S_n \text{ feasible}} \sum_i v_i(S)$$
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What is the right notion of equilibrium in a market?
Some Definitions

Definition
A set of prices $p$ together with an allocation $S_1, \ldots, S_n$ form an
($\epsilon$-approximate) *Walrasian equilibrium* if:

1. $S_1, \ldots, S_n$ is feasible, and
2. For all $i$, buyer $i$ is receiving his ($\epsilon$) most preferred bundle
given the prices:

   \[ v_i(S_i) - \sum_{j \in S_i} p_j \geq \max_{S^* \subseteq G} \left( v_i(S^*) - \sum_{j \in S^*} p_j \right) - \epsilon \]

   and,

3. All unallocated items have zero price: for all $j \notin S_1 \cup \ldots \cup S_n$, $p_j = 0$. 
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At Walrasian equilibrium, no buyer wants to buy a different bundle, and the seller does not want to lower any of the prices – the only things that aren’t selling can’t sell (they already have price 0).
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2. If so, are they compatible with social welfare maximization?
Theorem

If $S_1, \ldots, S_n$ form an $\epsilon$-Walrasian equilibrium allocation, then they achieve nearly optimal welfare. In particular:

$$\sum_i v_i(S_i) \geq \text{OPT} - \epsilon n$$
Proof

1. Let $p$ be the corresponding Walrasian equilibrium prices, and let $S'_1, \ldots, S'_n$ be any other feasible allocation.
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2. We know from the 2nd Walrasian equilibrium condition that for every player $i$, we have:

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3. Summing over buyers:

   $$\sum_i \left( v_i(S_i) - \sum_{j \in S_i} p_j \right) \geq \sum_i \left( v_i(S'_i) - \sum_{j \in S'_i} p_j \right) - \epsilon n$$
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4. Reordering:

$$\sum_i v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_i v_i(S'_i) - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j - \epsilon n$$
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2. Rewriting:

$$\sum_{i} v_i(S_i) \geq \sum_{i} v_i(S'_i) + (\sum_{j} p_j - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j) - \epsilon n \geq \sum_{i} v_i(S'_i) - \epsilon n$$
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3. Finally, taking \( S'_1, \ldots, S'_n \) to be the optimal allocation gives the theorem. (Tada!)
Walrasian Equilibrium are Great! Do They Exist?

1. We’ll start with a simple case: unit demand buyers (want to buy only 1 item):

\[ v_i(S) = \max_{j \in S} v_i(\{j\}) \]

We can think about such a valuation function as being determined by just \( m \) numbers, one for each good:

\[ v_{i,j} \equiv v_i(\{j\}) \leq 1 \]
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**Theorem**

*For any set of unit demand buyers, a Walrasian equilibrium always exists.*
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6. Deferred acceptance like...
Proof

Algorithm 1 The Ascending Price Auction with increment $\epsilon$.

For all $j \in G$, set $p_j = 0$, $\mu(j) = \emptyset$.

while There exist any unmatched bidders do

for Each unmatched bidder $i$ do

    $i$ “bids” on $j^* = \arg \max_j (v_{i,j} - p_j)$ if $v_{i,j^*} - p_{j^*} > 0$. Otherwise, bidder $i$ drops out. (and is “matched” to nothing): $\mu(j^*)$ is now unmatched. Set $\mu(j^*) \leftarrow i$

    $p_{j^*} \leftarrow p_{j^*} + \epsilon$

end for

end while

Output $(p, \mu)$. 
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6. (Lemma Tada!)
Proof

Lemma

The output \((p, \mu)\) of the ascending price auction is an \(\epsilon\)-approximate Walrasian equilibrium.
Proof

We’ll verify the 3 conditions:

1. By construction it outputs a feasible allocation.
2. If good $j$ is unallocated, it must never have received a bid in the auction, and hence $p_j = 0$.
3. Finally: $v_i,\mu(i) - p\mu(i) \geq \max_j (v_i, j - p_j) - \epsilon$. This is because...
4. at the time bidder $i$ was matched to good $\mu(i)$, we must have had: $\mu(i) \in \arg \max_j (v_i, j - p_j)$
5. Since that time $p_j$ increased by $\epsilon$, no other price has decreased.
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\text{for all } j \in S^*, \quad \mu(j) \leftarrow i, \quad p_j \leftarrow p_j + \frac{\epsilon}{m}.
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4. We can formalize this.
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1. For price vectors $p, p'$, write $p \leq p'$ to mean that $p_j \leq p'_j$ for all $j$. Let $w_i(p) = \arg \max_{S \subseteq G} (v_i(S) - \sum_{j \in S} p_j)$ be player $i$’s demand set at prices $p$.

Definition

Valuation function $v_i$ satisfies the gross substitutes property if for every $p \leq p'$ and for every $S \in w_i(p)$, if $S' = \{j \in S : p_j = p'_j\}$, then there exits $S^* \in w_i(p')$ such that $S' \subseteq S^*$.

In other words, “Raising the prices on goods $j \neq i$ doesn’t decrease a bidder’s demand for good $j$”.

2. This is what we need: Any good for which bidder $i$ has not been out-bid on has not had its price raised, and so must still be part of a bundle in bidder $i$’s demand set.

3. Hence, we have:

Theorem

In any market in which all buyers satisfy the gross substitutes condition, Walrasian equilibria exist.
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1. For price vectors $p, p'$, write $p \preceq p'$ to mean that $p_j \leq p'_j$ for all $j$. Let $w_i(p) = \arg\max_{S \subseteq G} \left( v_i(S) - \sum_{j \in S} p_j \right)$ be player $i$’s demand set at prices $p$.

Definition
Valuation function $v_i$ satisfies the gross substitutes property if for every $p \preceq p'$ and for every $S \in w_i(p)$, if $S' = \{j \in S : p_j = p'_j\}$, then there exits $S^* \in w_i(p')$ such that $S' \subseteq S^*$.
In other words, “Raising the prices on goods $j \neq i$ doesn’t decrease a bidder’s demand for good $j$”.

2. This is what we need: Any good for which bidder $i$ has not been out-bid on has not had its price raised, and so must still be part of a bundle in bidder $i$’s demand set.

3. Hence, we have:

Theorem

In any market in which all buyers satisfy the gross substitutes condition, Walrasian equilibria exist.
Thanks!

See you next class — stay healthy!