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Overview

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Overview

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- We will again prohibit the use of money...
- Used in practice to match medical students to residencies, pledges to sororities, students to public schools in various districts.
1. Let $M$ and $W$ denote sets of *students* and *schools* respectively. Assume $|M| = |W| = n$. 

A Model
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2. A Matching:

Definition
A matching $\mu : M \cup W \to M \cup W$ is an assignment of students to schools so that each student is assigned to exactly one school and vice versa. For each $m \in M$ and $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$. 
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3. Each $m \in M$ has a strict preference ordering $\succ_m$ over the set $W$, and each $w \in W$ has a strict preference ordering $\succ_w$ over the set $M$. 
Goals

▶ Just as in last lecture, we have two desiderate:

1. We would like the matching that we compute to be good in some sense, and
2. We would like to incentivize participants to reveal their true preferences to the mechanism.

▶ We’ll be able to find “good” matchings — and will have limited success managing preferences.
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We’ll be able to find “good” matchings — and will have limited success managing preferences.
What makes a Matching Reasonable

1. Minimal requirement: Stability. We can suggest the matching, but can’t force people into matchings.

   Definition
   A matching $\mu$ is unstable if there exists an $m \in M$ and $w \in W$ such that $\mu(m) \neq w$, but:
   $w \succ m \mu(m)$ and $m \succ w \mu(w)$

   We call such an $(m, w)$ pair a blocking pair for $\mu$. (A blocking pair witnesses instability because $m$ and $w$ could mutually benefit by leaving their proposed partners and pairing with one another).

   A matching $\mu$ is stable if it has no blocking pairs.
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We call such an \((m, w)\) pair a \textit{blocking pair} for \( \mu \). (A blocking pair witnesses instability because \( m \) and \( w \) could mutually benefit by leaving their proposed partners and pairing with one another).

A matching \( \mu \) is \textit{stable} if it has no blocking pairs.

3. We might more ambitiously want to compute the “best” stable matching – but do they even exist?
They Do Exist!

**Theorem (Gale and Shapley)**

*For any set of preferences \((\succ_m^1, \ldots, \succ_m^n, \succ_w^1, \ldots, \succ_w^n)\), a stable matching \(\mu\) exists.*
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**Theorem (Gale and Shapley)**

For any set of preferences $(\succeq_{m_1}, \ldots, \succeq_{m_n}, \succeq_{w_1}, \ldots, \succeq_{w_n})$, a stable matching $\mu$ exists.

1. An algorithmic proof: we’ll prove existence by showing how to find one.
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1. An algorithmic proof: we’ll prove existence by showing how to find one.
2. The student applying *deferred acceptance* algorithm.
Algorithm 1 The Deferred Acceptance Algorithm (Student Applying Version)

DeferredAcceptance($\succ$):

Initially, $\mu(m) = \emptyset$ for all $m \in M$. (i.e. nobody is yet matched).

Each student $m \in M$ applies to his most preferred $w \in W$. For each school $w \in W$, let $m'$ be its most preferred student among the set that applied to it, and set $\mu(m') \leftarrow w$. All other students are rejected (and hence unmatched).

while There exists any unmatched student $m \in M$: do

- $m$ applies to his most preferred $w \in W$ that he has not yet applied to.

- If $m \succ_w \mu(w)$, then $\mu(\mu(w)) \leftarrow \emptyset$ and $\mu(w) \leftarrow m$ (i.e. $w$ rejects its current match and instead matches to $m$). Else, $m$ is rejected.

end while

Return $\mu$
Proof

1. The algorithm halts: every school receives at least one application over the course of the algorithm. (If there is a school without an application, there is an unmatched student, and the algorithm has not halted unless he has applied to all schools). Once a school has received an application, it becomes matched, and stays matched for the rest of the algorithm.
Proof

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2. Since $|W| = |M|$, once all schools are matched, all students are matched.
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2. Since $|W| = |M|$, once all schools are matched, all students are matched.

3. So the algorithm halts after at most $n^2$ applications, since no student applies to the same school twice.
Proof

1. The final matching $\mu$ cannot have any blocking pairs.
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2. Suppose otherwise: there is a blocking pair $(m_1, w_1)$ with $\mu(m_1) \neq w_1$, but $w_1 \succ_{m_1} \mu(m_1)$ and $m_1 \succ_{w_1} \mu(w_1)$. 

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3. Since $w_1 \succ_{m_1} \mu(m_1)$, $m_1$ must have applied to $w_1$ before he applied to $\mu(m_1)$.
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4. Since $\mu(m_1) \neq w_1$, $m_1$ must have been rejected by $w_1$ in favor of some other student $m'$.
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4. Since $\mu(m_1) \neq w_1$, $m_1$ must have been rejected by $w_1$ in favor of some other student $m'$.
5. Since schools only ever change who they are matched to in favor of more preferred students, we must have:

$$\mu(w_1) \succeq_{w_1} m' \succ_{w_1} m_1$$

which contradicts $m_1 \succ_{w_1} \mu(w_1)$. 
Proof

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Good matchings?

1. What is a good matching? Not everyone can receive their favorite match.
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2. Define:

Definition
For \( m \in M \) and \( w \in W \), we say that \( w \) is achievable for \( m \) (and vice versa) if there exists a stable matching \( \mu \) such that \( \mu(m) = w \).
Good matchings?

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Definition
For \( m \in M \) and \( w \in W \), we say that \( w \) is achievable for \( m \) (and vice versa) if there exists a stable matching \( \mu \) such that \( \mu(m) = w \).

3. Optimality: The best among all achievable matchings:

Definition
A matching \( \mu \) is student optimal if for every achievable pair \((m, w)\), \( \mu(m) \succeq_m w \) Similarly, we can define school optimal matchings, and student and school pessimal matchings. (A matching \( \mu \) is school pessimal if for every achievable pair \((m, w)\), \( m \succeq_w \mu(w) \))
Its Good to be on the Applying Side

Theorem

The stable matching \( \mu \) output by the student-applying deferred acceptance algorithm is student optimal.
Proof

1. Suppose otherwise. There must be some first round $k$ at which a student $m$ is rejected by his most preferred achievable school $w$, in favor of $m'$. $m' \succ_w m$.

2. Since $w$ is achievable for $m$, there must be some stable matching $\mu$ such that $\mu(m) = w$ and $\mu(m') = w'$ (and hence $w'$ is achievable for $m'$).

3. We must have $w \succ m' w'$ (since $m'$ applied to $w$, and can't have been rejected by any achievable school since by assumption, $k$ was the first round at which a student was rejected by an achievable school.)

4. Combining: $m' \succ w m w \succ m' w'$

5. $(m', w)$ form a blocking pair for $\mu$, contradicting stability.

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$$m' \succ_w m \quad w \succ_{m'} w'$$

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Proof

1. Suppose otherwise. There must be some first round \( k \) at which a student \( m \) is rejected by his most preferred achievable school \( w \), in favor of \( m' \). \( m' \succ_w m \).

2. Since \( w \) is achievable for \( m \), there must be some stable matching \( \mu \) such that \( \mu(m) = w \) and \( \mu(m') = w' \) (and hence \( w' \) is achievable for \( m' \)).

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4. Combining:

\[
m' \succ_w m \quad w \succ_{m'} w'
\]

5. \( (m', w) \) form a blocking pair for \( \mu \), contradicting stability.

6. Tada!
Its Bad to be on the Receiving Side

**Theorem**

*The stable matching produced by the student-applying deferred acceptance algorithm is school pessimal.*
Proof

1. In fact: every student-optimal stable matching $\mu$ is school pessimal. Suppose otherwise.
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1. In fact: every student-optimal stable matching $\mu$ is school pessimal. Suppose otherwise.

2. There exists some $w$ with $\mu(w) = m$, and $m \succ_w m'$ for some other achievable student $m'$.

3. So there must exist a different stable matching $\mu'$ with $\mu'(m') = w$, and $\mu'(m) = w'$.

4. But we must have $w \succ m w' = \mu'(m)$ because $\mu$ is student-optimal and $w'$ is achievable for $m$.

5. So $(m, w)$ are a blocking pair for $\mu'$, which contradicts its stability.

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Proof

1. In fact: every student-optimal stable matching \( \mu \) is school pessimal. Suppose otherwise.

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5. So $(m, w)$ are a blocking pair for $\mu'$, which contradicts its stability.

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What about Incentives?

**Theorem**

The student applying deferred acceptance algorithm is dominant strategy incentive compatible for the students. (i.e. reporting their true preferences $\succ_m$ is a dominant strategy for each $m \in M$).
Proof

1. Suppose otherwise: there is a set of preferences
   \( \succ = (\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n}) \) and a deviation \( \succ'_{m_1} \) such
   that if \( \mu = \text{DE}(\succ) \) and \( \mu' = \text{DE}(\succ') \) (where
   \( \succ' = (\succ'_{m_1}, \succ_{-m_1}) \)), then:

   \[
   \mu'(m_1) \succ_{m_1} \mu(m_1).
   \]
Proof

1. Suppose otherwise: there is a set of preferences \( \succ = (\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n}) \) and a deviation \( \succ'_{m_1} \) such that if \( \mu = DE(\succ) \) and \( \mu' = DE(\succ') \) (where \( \succ' = (\succ'_{m_1}, \succ_{-m_1}) \)), then:

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\mu'(m_1) \succ_{m_1} \mu(m_1).
\]

2. We know that \( \mu \) is stable and student optimal with respect to preferences \( \succ \), and \( \mu' \) is stable and student optimal with respect to preferences \( \succ' \).
Proof

1. Suppose otherwise: there is a set of preferences
\( \succ = (\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n}) \) and a deviation \( \succ_{m_1}' \) such that if \( \mu = DE(\succ) \) and \( \mu' = DE(\succ') \) (where \( \succ' = (\succ_{m_1}', \succ_{-m_1}) \)), then:
\[ \mu'(m_1) \succ_{m_1} \mu(m_1). \]

2. We know that \( \mu \) is stable and student optimal with respect to preferences \( \succ \), and \( \mu' \) is stable and student optimal with respect to preferences \( \succ' \).

3. Define two sets:

\[ R = \{ m : \mu'(m) \succ_m \mu(m) \} \]
\[ T = \{ w : \mu'(w) \in R \} \]
Proof

1. Suppose otherwise: there is a set of preferences $\succ = (\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n})$ and a deviation $\succ'_{m_1}$ such that if $\mu = DE(\succ)$ and $\mu' = DE(\succ')$ (where $\succ' = (\succ'_{m_1}, \succ'_{-m_1})$), then:

$$\mu'(m_1) \succ_{m_1} m(m_1).$$

2. We know that $\mu$ is stable and student optimal with respect to preferences $\succ$, and $\mu'$ is stable and student optimal with respect to preferences $\succ'$

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   3.1 The set of students who prefer $\mu'$ to $\mu$:

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1. Suppose otherwise: there is a set of preferences \( \succ = (\succ_{m_1}, \ldots, \succ_{m_n}, \succ_{w_1}, \ldots, \succ_{w_n}) \) and a deviation \( \succ'_{m_1} \) such that if \( \mu = DE(\succ) \) and \( \mu' = DE(\succ') \) (where \( \succ' = (\succ'_{m_1}, \succ_{-m_1}) \)), then:
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3. Define two sets:
   3.1 The set of students who prefer \( \mu' \) to \( \mu \):
   \[
   R = \{ m : \mu'(m) \succ_{m} \mu(m) \}
   \]
   3.2 The set of schools whose matches in \( \mu' \) are in \( R \) (and so prefer them to their match in \( \mu \)):
   \[
   T = \{ w : \mu'(w) \in R \} \]
Proof

1. We will show:

   1.1 \( w \in T \iff \mu(w) \in R \). (i.e. if a school's partner in \( \mu' \) prefers \( \mu' \) to \( \mu \), so does its partner in \( \mu \)), and from this derive that:

   1.2 There exists a \( w \ell \in T \) and a \( m_r \in R \) such that \((w \ell, m_r)\) form a blocking pair in \( \mu' \) with respect to \( \succ' \), a contradiction.

2. We'll start with the first claim...
1. We will show:
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1. For any \( m \in R \), let \( w = \mu'(m) \in T \). Let \( m' = \mu(w) \) be \( w \)'s partner in \( \mu \).
Proof

Claim

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1. For any \( m \in R \), let \( w = \mu'(m) \in T \). Let \( m' = \mu(w) \) be \( w \)'s partner in \( \mu \).

2. If \( m' = m_1 \), we are done. Otherwise we can assume \( m' \neq m_1 \), and therefore that \( \succ_m = \succ'_m \).
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3. Since \( m \in R \), we know that: \( w = \mu'(m) \succ_m \mu(m) \).
Proof

Claim

\[ w \in T \iff \mu(w) \in R \]

1. For any \( m \in R \), let \( w = \mu'(m) \in T \). Let \( m' = \mu(w) \) be \( w \)'s partner in \( \mu \).
2. If \( m' = m_1 \), we are done. Otherwise we can assume \( m' \neq m_1 \), and therefore that \( \succsim_{m'} = \succsim_{m}' \).
3. Since \( m \in R \), we know that: \( w = \mu'(m) \succsim_m \mu(m) \).
4. Since \( \mu \) is stable w.r.t \( \succsim \), it must be that \( \mu(w) = m' \succsim_w m \).
Proof

Claim

\[ w \in T \iff \mu(w) \in R \]

1. For any \( m \in R \), let \( w = \mu'(m) \in T \). Let \( m' = \mu(w) \) be \( w \)'s partner in \( \mu \).

2. If \( m' = m_1 \), we are done. Otherwise we can assume \( m' \neq m_1 \), and therefore that \( \succ_m = \succ'_m \).

3. Since \( m \in R \), we know that: \( w = \mu'(m) \succ_m \mu(m) \).

4. Since \( \mu \) is stable w.r.t \( \succ \), it must be that \( \mu(w) = m' \succ_w m \).

5. Because \( \mu' \) is stable w.r.t. \( \succ' \), it must be that \( \mu'(m') \succ_{m'} \mu(m') = w \).
Proof

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6. Hence \( m' \in R \) as we wanted
Proof

Claim
There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$

2. So when running DE($\succ$), it must be that every $m \in R$ applies to $\mu'(m)$, and is rejected by $\mu'(m)$ at some round.

3. Let $m_\ell$ be the last $m \in R$ who applies during the DE algorithm. This application must be to $\mu(m_\ell) \equiv w_\ell$.

4. By the first claim, since $m_\ell \in R$, $w_\ell \in T$.

5. It must be that $w_\ell$ rejected $\mu'(w_\ell)$ at a strictly earlier round (since $m_\ell$ is the last $m \in R$ to apply), and hence when $m_\ell$ applies to $w_\ell$, $w_\ell$ rejects some $m_r \not\in R$ such that:

$$m_r \succ w_\ell \mu'(w_\ell)$$
Proof

Claim
There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$

1. Since for every $m \in R$, $\mu'(m) \succ_m \mu(m)$, by stability, it must be that for all $w \in T$: $\mu(w) \succ_w \mu'(w)$. 
Proof

Claim
There exists a \( w_\ell \in T \) and a \( m_r \in R \) such that \((w_\ell, m_r)\) form a blocking pair in \( \mu' \) with respect to \( \succ' \)

1. Since for every \( m \in R \), \( \mu'(m) \succ_m \mu(m) \), by stability, it must be that for all \( w \in T \): \( \mu(w) \succ_w \mu'(w) \).

2. So when running DE(\( \succ \)), it must be that every \( m \in R \) applies to \( \mu'(m) \), and is rejected by \( \mu'(m) \) at some round.
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There exists a $w_\ell \in T$ and a $m_r \in R$ such that $(w_\ell, m_r)$ form a blocking pair in $\mu'$ with respect to $\succ'$

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There exists a \( w_\ell \in T \) and a \( m_r \in R \) such that \((w_\ell, m_r)\) form a blocking pair in \( \mu' \) with respect to \( \succ' \)

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2. So when running \( \text{DE}(\succ') \), it must be that every \( m \in R \) applies to \( \mu'(m) \), and is rejected by \( \mu'(m) \) at some round.
3. Let \( m_\ell \) be the last \( m \in R \) who applies during the \( \text{DE} \) algorithm. This application must be to \( \mu(m_\ell) \equiv w_\ell \).
4. By the first claim, since \( m_\ell \in R, w_\ell \in T \).
5. It must be that \( w_\ell \) rejected \( \mu'(w_\ell) \) at a strictly earlier round (since \( m_\ell \) is the last \( m \in R \) to apply), and hence when \( m_\ell \) applies to \( w_\ell \), \( w_\ell \) rejects some \( m_r \not\in R \) such that: \( m_r \succ_{w_\ell} \mu'(w_\ell) \)
Proof

$$m_r \succ_{w_{\ell}} \mu'(w_{\ell})$$
Proof

\[ m_r \succ_w \mu'(w) \]

1. Since \( m_r \) had applied to \( w \) before \( \mu(m_r) \), it must be that:

\[ w \succ_m \mu(m_r) \]
Proof

\[ m_r \succ_{w_\ell} \mu'(w_\ell) \]

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3. Together with the above, this means \((m_r, w_\ell)\) form a blocking pair for \( \mu' \), a contradiction.

4. Tada!
Proof

\[ m_r \succ_{w_\ell} \mu'(w_\ell) \]

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Proof

\[ m_r \succeq_{w_\ell} \mu'(w_\ell) \]

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3. Together with the above, this means \((m_r, w_\ell)\) form a blocking pair for \( \mu' \), a contradiction.

4. Tada!
Thanks!

See you next class — stay healthy!