Posted Pricings and Prophet Inequalities

Aaron Roth

University of Pennsylvania

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Overview

▶ We’ve seen (if we know the valuation distributions) how to maximize social welfare and revenue with an auction.

▶ But auctions are difficult to run. They require e.g. all participants to be present and coordinating.

▶ Many things are instead sold via posted prices.

▶ This lecture: How to approximate the welfare and revenue of the optimal auction with posted prices.
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- But auctions are difficult to run. They require e.g. all participants to be present and coordinating.
- Many things are instead sold via posted prices.
- This lecture: How to approximate the welfare and revenue of the optimal auction with posted prices.
Pricing for a single item (e.g. a car)

A Model:

- $k$ recognizable types of buyers (based on demographics, purchase history, or anything else).

- Buyers of type $i$ have valuation $v_i \sim D_i$, where $D_i$ is regular.

- Buyers arrive one at a time until the item is sold.

- Buyers of type $i$ face price $p_i$. If $v_i \geq p_i$ they buy the item, and we get revenue $p_i$. Otherwise they pass.

Are there choices of $p_i$ that allow us to approximate the welfare or revenue of the optimal auction?
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Consider the following game:

- In each of $n$ steps $i \in \{1, \ldots, n\}$, you are offered a prize $\pi_i \sim G_i$. (The distributions $G_i$ are known in advance).

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- A *prophet* could foresee all of the prizes and make sure to always take the highest one. His expected profit would be:

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\text{Profit(Prophet)} = \mathbb{E}[\max_i \pi_i]
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Prophets and Profits (an Interlude)

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Theorem
*For every set of distributions $G_1, \ldots, G_n$, there is a threshold strategy that guarantees reward at least $\frac{1}{2} \mathbb{E}[\max_i \pi_i]$.**
The Prophet Inequality

- Notation: $z^+ = \max(z, 0)$, $V^* = \max_i \pi_i$.

- We'll use threshold $t = \frac{1}{2} \mathbb{E}[V^*]$.

- We'll use language of the economic application:
  - "item is unsold" ⇔ "We don't accept any prizes"
  - "item is sold" ⇔ "We accept a prize"

- We'll prove the prophet inequality by decomposing expected reward between:
  1. Expected revenue, and
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- Strategy: Prove lower bounds on expected revenue and buyer utility separately.
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- Expected Revenue:

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E[\text{Revenue}] = p \cdot \Pr[\text{Item is sold}] = \frac{1}{2}E[V^*] \cdot \Pr[\text{Item is sold}]
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- Buyer Utility:

\[ E[\text{Utility}] = \sum_{i=1}^{n} E[(v_i - p)] \cdot \Pr[\text{Item is unsold before } i] \geq \sum_{i=1}^{n} E[(v_i - p)] \cdot \Pr[\text{Item is unsold}] \geq E[\max_i (v_i - p)] \cdot \Pr[\text{Item is unsold}] = \frac{1}{2} E[V^*] \cdot \Pr[\text{Item is unsold}] \]
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Immediate implications for welfare maximization!

- Using a *single* fixed price $p = \frac{1}{2}E[V^*]$, can obtain half the expected welfare of the VCG mechanism.
Welfare

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- Without needing to gather all bidders ahead of time, and despite the uncertainty about realizations!
Welfare

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- Without needing to gather all bidders ahead of time, and despite the uncertainty about realizations!
- What about for revenue?
Revenue

Recall that for monotone allocation rules $X$ paired with truthful pricings $P$:

$$E[\text{Revenue}] = E\left[\sum_{i=1}^{n} \phi_i(v_i)X(v)\right]$$

Optimal revenue is $\text{OPT} = E[\max_i(\phi_i(v_i))]$.

Define $\pi_i = (\phi_i(v_i))$. So $E[V^*] = \text{OPT}$.

We can achieve virtual value at least $\frac{1}{2}\text{OPT}$ with threshold $t = \frac{\text{OPT}}{2}$.

This corresponds to setting threshold/price $p_i = \phi_i - \frac{\text{OPT}}{2}$.

(Note a fixed price corresponds to a monotone allocation rule with payment = price)

We need to use different prices for different types of bidders, but approximate optimal revenue.
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- We can achieve virtual value at least $\frac{1}{2}\text{OPT}$ with threshold $t = \text{OPT}/2$. 

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- Define \( \pi_i = (\phi_i(v_i))^+ \). So \( E[V^*] = \text{OPT} \).
- We can achieve \textit{virtual value} at least \( \frac{1}{2} \text{OPT} \) with threshold \( t = \text{OPT}/2 \).
- This corresponds to setting threshold/price \( p_i = \phi_i^{-1}(\frac{\text{OPT}}{2}) \).
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- We need to use different prices for different types of bidders, but approximate optimal revenue.
Thanks!

See you next class — stay healthy!