Zero Sum Games and the Minimax Theorem

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University of Pennsylvania

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Overview

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- Playing the polynomial weights algorithm in a zero sum game leads to equilibrium (a plausible dynamic!)
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- They have a very special property: the minimax theorem.
- And a close connection to the polynomial weights algorithm (and related algorithms)
- Playing the polynomial weights algorithm in a zero sum game leads to equilibrium (a plausible dynamic!)
- In fact, we’ll use it to prove the minimax theorem.
Zero Sum Games

Definition
A two player zero sum game is any two player game such that for every \( a \in A_1 \times A_2 \), \( u_1(a) = -u_2(a) \).(i.e. at every action profile, the utilities sum to zero)
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1. Strictly adversarial games: The only way for player 1 to improve his payoff is to harm player 2, and vice versa.
2. Closely related to linear programming, adversarial machine learning, and lots of other things.
Consider the “Presidential Election Game”:

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Since utilities always sum to zero, we can more economically write the game by specifying only the row player’s utility.
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The row player (Max) wishes to maximize the utility. The column player (Min) wishes to minimize the utility.
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3. Min should pick the action that minimizes her cost! She can compute:

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E[\text{Morality}] = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot (-2) = \frac{1}{2}
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E[\text{Tax - Cuts}] = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0
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So she plays Tax-cuts.
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2. Min will play the strategy that minimizes her cost:

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5. And if Min goes first, she should play:
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3p - 2(1 - p) = -p + (1 - p) \iff 5p - 2 = 1 - 2p \iff p = \frac{3}{7}
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5. And when Min best responds, he gets payoff \(1 - 2p = 1/7\).
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4. Lets investigate further...
Order of Play

We use the notation \([n] = \{1, 2, \ldots, n\}\), and \(\Delta[n]\) to denote the set of probability distributions over \([n]\):
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**Definition**

For an \(n \times m\) matrix \(U\) (think about this as the payoff matrix in a two player zero sum game if you like):

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\max \min (U) = \max_{p \in \Delta[n]} \min_{y \in [m]} \sum_{i=1}^{n} p_i \cdot U(i, y)
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If \(U\) is a zero sum game, then \(\max \min (U)\) represents the payoff that Max can guarantee if he goes first, and \(\min \max (U)\) represents the payoff that he can guarantee if Min goes first.
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If \(U\) is a zero sum game, then \(\max \min(U)\) represents the payoff that Max can guarantee if he goes first, and \(\min \max(U)\) represents the payoff that he can guarantee if Min goes first.
The Minimax Theorem

Recall going first is not an advantage. In math, for any game $U$:

$\min \max(U) \geq \max \min(U)$
The Minimax Theorem

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It turns out that in a zero sum game, it is also not a disadvantage:

**Theorem (Von Neumann)**

*In any zero sum game $U$:

$$\min \max(U) = \max \min(U)$$

**Corollary**

In any Nash equilibrium of a zero sum game, Max plays a maxmin strategy and Min plays a minmax strategy. Note that these can be computed without needing to reason about what the other player is doing.

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All Nash equilibria in Zero sum games have the same payoff – the $\max \min$ value of the game.
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2. It means Zero-sum games are easy to play: no need for counter-speculation.
The Minimax Theorem

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2. It means Zero-sum games are easy to play: no need for counter-speculation.

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4. Previously, Borell had proven it for the special case of $5 \times 5$ matrices, and thought it was false for larger matrices.
5. But well give an easy, constructive proof.
1. Suppose the theorem were false: there is some game $U$ for which $\min \max(U) > \max \min(U)$. 

2. Write $v_1 = \min \max(U)$ and $v_2 = \max \min(U)$ (And so $v_1 = v_2 + \epsilon$ for some constant $\epsilon > 0$).

3. In other words: if Min has to go first, then Max can guarantee payoff at least $v_1$, but if Max is forced to go first, then Min can force Max to have payoff only $v_2$. 

Proof
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3. In other words: if Min has to go first, then Max can guarantee payoff at least $\nu_1$, but if Max is forced to go first, then Min can force Max to have payoff only $\nu_2$. 
Proof: A Thought Experiment

Let's consider what happens when Min and Max repeatedly play against each other as follows, for $T$ rounds:

1. Min will play using the polynomial weights algorithm. i.e. at each round $t$, the weights $w^t$ of the polynomial weights algorithm will form her mixed strategy, and she will sample an action at random from this distribution, updating based on the losses she experiences at that round.

2. Max will play the best response to Min's strategy. i.e. Max will play $x^t = \text{arg max}_x E_{y \sim w^t}[U(x, y)]$. 

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What we know about each player’s average payoffs when they play in this manner?
Proof: A Thought Experiment

We know from the guarantee of the polynomial weights algorithm:

$T^T X_t = 1 \sum_{x_t} U(x_t, y_t) \leq T^{\min} y^* T^T X_t = 1 \sum_{x_t} U(x_t, y^*) + \Delta(T)$

$= \min_{y^*} T^T X_t = 1 \sum_{x_t} U(x_t, y^*) + \Delta(T)$

$\overline{x}$ is the mixed strategy that puts weight $1/T$ on each action.

$\Delta(T)$ is the regret bound of the polynomial weights algorithm:

$\Delta(T) = 2r \log n T$

By definition, $\min_{y^*} \sum_{x_t} \overline{x} U(x_t, y^*) \leq \max \min U(x_t, \cdot) = v_2$ and so:

$T^T X_t = 1 \sum_{x_t} U(x_t, y_t) \leq v_2 + \Delta(T)$
Proof: A Thought Experiment

We know from the guarantee of the polynomial weights algorithm:

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[U(x^t, y^t)] \leq \frac{1}{T} \min_{y^*} \sum_{t=1}^{T} U(x^t, y^*) + \Delta(T)
\]

\(\bar{x}\) is the mixed strategy that puts weight 1

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\[
= \min_{y^*} \mathbb{E}_{x \sim \bar{x}} [U(x, y^*)] + \Delta(T)
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\(\bar{x}\) is the mixed strategy that puts weight \(1/T\) on each action \(x^t\). 
\(\Delta(T)\) is the regret bound of the polynomial weights algorithm:
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\(\bar{x}\) is the mixed strategy that puts weight 1/\(T\) on each action \(x^t\).
\(\Delta(T)\) is the regret bound of the polynomial weights algorithm:

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\Delta(T) = 2\sqrt{\frac{\log n}{T}}.
\]
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We know from the guarantee of the polynomial weights algorithm:

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\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[U(x^t, y^t)] \leq \frac{1}{T} \min_{y^*} \sum_{t=1}^{T} U(x^t, y^*) + \Delta(T)
\]

\[
= \min_{y^*} \sum_{t=1}^{T} \frac{1}{T} U(x^t, y^*) + \Delta(T)
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\[
= \min_{y^*} \mathbb{E}_{x \sim \tilde{x}} [U(x, y^*)] + \Delta(T)
\]

\(\tilde{x}\) is the mixed strategy that puts weight 1/T on each action \(x^t\). \(\Delta(T)\) is the regret bound of the polynomial weights algorithm:

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By definition, \(\min_{y^*} \mathbb{E}_{x \sim \tilde{x}} U(x, y^*) \leq \max \min(U) = v_2\) and so:

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\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[U(x^t, y^t)] \leq v_2 + \Delta(T)
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Proof: A Thought Experiment

But: on each day \( t \) Max is best responding to Min’s mixed strategy \( w^t \). So...
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Combining these inequalities:

$$v_1 \leq v_2 + \Delta(T)$$
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Since $v_1 = v_2 + \epsilon$:

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Since \( v_1 = v_2 + \epsilon \):

\[
\epsilon \leq \Delta(T)
\]

Taking \( T \) large enough leads to contradiction.
Reflection

1. An amazing feature of Polynomial Weights: It guarantees that no matter what, you do as well as if you had gotten to observe your opponent’s strategy, and then best respond after the fact.
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4. The only information needed is the realized payoffs are for the actions as it plays the game.
Thanks!

See you next class — stay healthy!