Mechanism Design via Differential Privacy

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Overview

▸ We’ll have one final lecture using digital goods auctions as a testbed for techniques in mechanism design.

Recall two lectures ago: we designed a dominant strategy truthful auction that for any vector of valuations $v$, obtained revenue:

$$\text{Rev}(v) \geq \text{OPT}(v) - O(\sqrt{n})$$

where $\text{OPT}(v) = \max_{p \in [0, 1]} p \cdot \{|i| : v_i \geq p\}$.

This class: we will relax our solution concept to asymptotic dominant strategy truthfulness, and try to obtain a better revenue guarantee.

Our tool: Differential privacy, a technique developed for protecting user privacy in data analysis.
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  where \( \text{OPT}_\nu = \max_{p \in [0,1]} p \cdot \left| \{ i : \nu_i \geq p \} \right| \).
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- This class: we will relax our solution concept to asymptotic dominant strategy truthfulness, and try to obtain a better revenue guarantee.
- Our tool: Differential privacy, a technique developed for protecting user privacy in data analysis.
Recall: Why can we not simply compute the price 
\[ p^* = \arg \max_{p \in [0,1]} p \cdot \{ i : v_i \geq p \} \] and charge that?

This price will be highly manipulable by (at least) one of the bidders – 
\[ p^* = v_i^* \] for some bidder \( i^* \), who will have strong incentive to change his bid.

So to get dominant strategy truthfulness, we needed to compute prices that were independent of bidder reports.

But what if we could compute a price \( p \) that is almost independent of the reported valuation \( v_i \) for every buyer \( i \)?

Will this yield in some sense an approximate truthfulness guarantee? This will be the idea behind our approach.
Approach

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▶ But what if we could compute a price $p$ that is almost independent of the reported valuation $v_i$ for every buyer $i$?
▶ Will this yield in some sense an approximate truthfulness guarantee? This will be the idea behind our approach.
Privacy Definitions

Definition
Two bid vectors $v, v' \in [0, 1]^n$ are *neighbors* if they differ in just a single agent’s bid: i.e. if there exists an index $i$ such that $v_j = v'_j$ for every index $j \neq i$. 

We can now define differential privacy:

**Definition**
A mechanism $M : [0, 1]^n \rightarrow O$ is $\epsilon$-differentially private if for every pair of neighboring bid vectors $v, v' \in [0, 1]^n$, and for every outcome $x \in O$:

$$\Pr[M(v) = x] \leq \exp(\epsilon) \Pr[M(v') = x].$$

Here you should think of $\epsilon < 1$ as a small constant, and think of $\exp(\epsilon) \approx 1 + \epsilon$. For $\epsilon \leq 1$ we have:

$$1 + \epsilon \leq \exp(\epsilon) \leq 1 + 2\epsilon.$$
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Approximate Truthfulness

We can also define what we mean by *approximate* dominant strategy truthfulness:

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\text{Definition } M : [0, 1]^n \to O \text{ is } \epsilon \text{-approximately dominant strategy truthful if for every bidder } i, \text{ every utility function } u_i : [0, 1] \times O \to [0, 1], \text{ every vector of valuations } v \in [0, 1]^n, \text{ and every deviation } v'_i \in [0, 1], \text{ if we write } v'_i = (v - i, v'_i), \text{ then:}
\]

\[
E_{o \sim M(v)}[u_i(v, o)] \geq E_{o \sim M(v')}[u_i(v, o)] - \epsilon
\]

In other words, we require that no bidder can substantially (by more than \( \epsilon \)) increase his utility by mis-reporting his valuation.
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**Definition**
A mechanism $M : [0, 1]^n \to \mathcal{O}$ is $\epsilon$-approximately dominant strategy truthful if for every bidder $i$, every utility function $u_i : [0, 1] \times \mathcal{O} \to [0, 1]$, every vector of valuations $\nu \in [0, 1]^n$, and every deviation $\nu_i' \in [0, 1]$, if we write $\nu' = (\nu_{-i}, \nu_i')$, then:

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E_{o \sim M(\nu)}[u_i(\nu_i, o)] \geq E_{o \sim M(\nu')}[u_i(\nu_i, o)] - \epsilon
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The Connection

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**Theorem**

*If a mechanism $M : [0, 1]^n \to \mathcal{O}$ is $\epsilon$-differentially private, then $M$ is also $\epsilon$-approximately dominant strategy truthful.*
Fix any buyer $i$, valuation vector $v$, and utility function $u_i : [0, 1] \times \mathcal{O} \to [0, 1]$. 
The Connection: Proof

Fix any buyer $i$, valuation vector $v$, and utility function $u_i : [0, 1] \times \mathcal{O} \rightarrow [0, 1]$.

$$E_{o \sim M(v)}[u_i(v_i, o)] = \sum_{o \in \mathcal{O}} u_i(v_i, o) \cdot \Pr[M(v) = o]$$
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\geq \sum_{o \in \mathcal{O}} u_i(v_i, o) \cdot \exp(-\epsilon) \Pr[M(v') = o]
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\geq \mathbb{E}_{o \sim M(v')}[u_i(v_i, o)] - \epsilon
\]

The last inequality follows because for $\epsilon < 1$, $\exp(-\epsilon) \geq 1 - \epsilon$, and $u_i(v_i, o) \leq 1$. 
So: to design an approximately truthful mechanism that guarantees high revenue, it is sufficient to design a differentially private mechanism with high revenue.
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Lets try a straightforward approach: directly picking a price that approximately maximizes revenue for the reported bidder valuations.
Exploiting the Connection

- So: to design an approximately truthful mechanism that guarantees high revenue, it is sufficient to design a differentially private mechanism with high revenue.
- Let's try a straightforward approach: directly picking a price that approximately maximizes revenue for the reported bidder valuations.
- As in the last two lectures, let's pick a finite subset of prices $P \subset [0, 1]$ to select from.
Consider the following mechanism (an instantiation of what is called “the exponential mechanism” in its more general form):
Consider the following mechanism (an instantiation of what is called “the exponential mechanism” in its more general form): \textbf{ExpMech}(v, \epsilon, P):

Define \( \text{Rev}(p, v) = p \cdot |\{ i : v_i \geq p \}|. \)

Output each \( p \in P \) according to the following probability distribution:

\[
\Pr[p] = \frac{1}{\phi(v)} \exp \left( \frac{\epsilon \cdot \text{Rev}(p, v)}{2} \right)
\]

where

\[
\phi(v) = \sum_{p \in P} \exp \left( \frac{\epsilon \cdot \text{Rev}(p, v)}{2} \right)
\]
Theorem

For any $\epsilon$, $P$: $\text{ExpMech}(\cdot, \epsilon, P)$ is $\epsilon$-differentially private.
Privacy/Truthfulness

Theorem
For any $\epsilon, P$: $\text{ExpMech}(. , \epsilon, P)$ is $\epsilon$-differentially private. (and thus $\epsilon$-approximately dominant strategy truthful)
Proof

Fix any pair of neighboring bid vectors $v, v'$ and any output $p$. We have:

\[
\Pr[\text{EM}(v, \epsilon, P) = p] = \phi(v) \exp(\epsilon \cdot \text{Rev}(p, v))^2 \leq \phi(v) \exp(\epsilon \cdot (\text{Rev}(p, v') + 1))^2 = \exp(\epsilon^2) \Pr[\text{EM}(v', \epsilon, P) = p].
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Proof

Fix any pair of neighboring bid vectors $v, v'$ and any output $p$. We have:

$$\Pr[EM(v, \epsilon, P) = p] = \frac{1}{\phi(v)} \exp \left( \frac{\epsilon \cdot Rev(p, v)}{2} \right)$$
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\leq \frac{1}{\phi(v)} \exp \left( \frac{\epsilon \cdot (Rev(p, v') + 1)}{2} \right)
= \frac{1}{\phi(v)} \exp \left( \frac{\epsilon}{2} \right) \exp \left( \frac{\epsilon \cdot Rev(p, v')}{2} \right)
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Proof

Fix any pair of neighboring bid vectors $v, v'$ and any output $p$. We have:

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$$\leq \exp \left( \frac{\epsilon}{2} \right) \frac{1}{\phi(v')} \exp \left( \frac{\epsilon}{2} \right) \exp \left( \frac{\epsilon \cdot \text{Rev}(p, v')}{2} \right)$$

$$= \exp(\epsilon) \Pr[EM(v', \epsilon, P) = p]$$
Theorem

For any $P$, $v$, $\epsilon$, $\delta$, with probability $1 - \delta$, $\text{ExpMech}(v, \epsilon, P)$ outputs a price $p$ such that:

$$\text{Rev}(p, v) \geq \max_{p^* \in P} \text{Rev}(p^*, v) - \frac{2}{\epsilon} \cdot \ln \left( \frac{|P|}{\delta} \right)$$
Proof

Let \( p^* = \max_{p^* \in P} \text{Rev}(p^*, \nu) \). For any value \( x \), we have:

\[
\Pr_{p}[\text{Rev}(p, \nu) \leq x] \leq \Pr_{p}[\text{Rev}(p, \nu) = \text{Rev}(p^*, \nu)] \leq |P| \cdot \exp(\epsilon \cdot \text{Rev}(p^*, \nu)/2) \leq |P| \cdot \exp(-\ln(|P|\delta)) = |P| \cdot \delta.
\]
Proof

Let \( p^* = \max_{p^* \in P} \text{Rev}(p^*, v) \). For any value \( x \), we have:

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\Pr_p[\text{Rev}(p, v) \leq x] \leq \frac{\Pr_p[\text{Rev}(p, v) \leq x]}{\Pr_p[\text{Rev}(p, v) = \text{Rev}(p^*, v)]}
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\]
Proof

Let $p^* = \max_{p^* \in P} Rev(p^*, v)$. For any value $x$, we have:

$$\Pr_p[Rev(p, v) \leq x] \leq \frac{\Pr_p[Rev(p, v) \leq x]}{\Pr_p[Rev(p, v) = Rev(p^*, v)]}$$

$$\leq \frac{|P| \cdot \exp(\epsilon x / 2)}{\exp(\epsilon Rev(p^*, v) / 2)}$$

$$= |P| \cdot \exp \left( \frac{\epsilon \cdot (x - Rev(p^*, v))}{2} \right)$$
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Now choose $x = \text{Rev}(p^*, \nu) - \frac{2}{\epsilon} \cdot \ln \left( \frac{|P|}{\delta} \right)$. Plugging that in above, we get:
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\Pr_p[\text{Rev}(p, v) \leq x] \leq |P| \cdot \exp \left( - \ln \left( \frac{|P|}{\delta} \right) \right) = |P| \cdot \frac{\delta}{|P|}
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$$= \delta$$
Putting it all Together

- We have: an approximately truthful way to select a revenue maximizing price from a finite set of prices $P$, with revenue guarantees with respect to the best price in $P$ that degrade with $|P|$. 

- This is a familiar tradeoff.

- Now our dependence on $|P|$ is only logarithmic...

- Let's again see what happens when we take the natural discretization: $P = \{\alpha, 2\alpha, 3\alpha, \ldots, \frac{1}{1/\alpha}\}$.

- Just as before, $|P| = \frac{1}{\alpha}$, and that we have the guarantee that for all $v$:

  $$\max_{p \in P} Rev(p, v) \geq \max_{p \in [0, 1]} Rev(p, v) - \alpha n$$
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- Just as before, $|P| = 1/\alpha$, and that we have the guarantee that for all $v$:

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Putting it all Together

- Combining our bounds we see that if we discretize the price space by $\alpha$, with probability 0.99, we obtain revenue:

$$\text{Rev}(p, v) \geq \text{OPT} - \alpha \cdot n - O \left( \frac{1}{\epsilon} \ln \left( \frac{1}{\alpha} \right) \right)$$
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- Choosing $\alpha = 1/n$, we find that for any $\epsilon$, we can obtain an $\epsilon$-approximately dominant strategy truthful mechanism which obtains revenue:

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- Choosing $\alpha = 1/n$, we find that for any $\epsilon$, we can obtain an $\epsilon$-approximately dominant strategy truthful mechanism which obtains revenue:

$$\text{Rev}(p, v) \geq \text{OPT} - O \left( \frac{\log n}{\epsilon} \right)$$

- If we take e.g. $\epsilon = O(1/\log(n))$, then we have an asymptotically truthful mechanism (in the sense that it becomes exactly truthful as $n \to \infty$) that improves by an exponential factor on the revenue guarantee that we were able to obtain with an exactly truthful mechanism for the same problem.
Thanks!

See you next class — stay healthy, and get vaccinated!