Maximizing Revenue in Expectation

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► But many auctions have a more pecuniary goal. What if we want to maximize revenue?
► What does that mean? What is our benchmark?
► This lecture: a case study for single item auctions.
Reasonable Benchmarks?

- The VCG mechanism was remarkable: we could always maximize welfare \textit{ex-post}.

- Revenue is \( p \) if \( v_i \geq p \), 0 otherwise.

- So ex-post, the revenue-optimal auction sets \( p = v_i \).

- But ex-ante, we don't have enough information.
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- So \textit{ex-post}, the revenue-optimal auction sets \( p = v_i \)... But \textit{ex-ante}, we don't have enough information.
The Average Case

- Suppose we know that bidders have valuations $v_i \sim D$ for some distribution $D$. 

  - For example, if $D$ is uniform on $[0, 1]$, then $F(p) = p$ and:

    $\max_p \text{Rev}(p) = \frac{1}{2} \cdot (1 - \frac{1}{2}) = \frac{1}{4}$
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$$Rev(p) = p \cdot (1 - F(p))$$

Where $F(p) = \Pr_{v \sim D}[v \leq p]$. 
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E_{v \sim D^n} \left[ \sum_{i=1}^{n} P_i(v) \right]
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- For truthfulness, we need \(X\) to be monotone non-decreasing...
- And we know:

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P_i(v) = v_i \cdot X_i(v) - \int_{0}^{v_i} X_i(z, v_{-i}) \, dz
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Myserson Optimal Auctions

- Lets assume monotonicity for now, and use our expression for $P$ to derive the optimal $X$. 

- If we are lucky and derive a monotone $X$, we will be done!

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So: We want to maximize

$$E_{v \sim D^n} \left[ \sum_{i=1}^{n} \phi(v_i) \cdot X(v) \right]$$

$$\phi(v_i) = \left( v_i - \frac{1 - F(v_i))}{f(v_i)} \right)$$

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- (Pointwise) optimal allocation rule: Give the item to the bidder \( i \) with highest \( \phi(v_i) \) if it’s positive. Otherwise give the item to nobody.
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- Our objective looks just like welfare with values replaced by virtual values.
- (Pointwise) optimal allocation rule: Give the item to the bidder \(i\) with highest \(\phi(v_i)\) if it's positive. Otherwise give the item to nobody.
- This is a monotone allocation rule if \(D\) is regular: \(\phi(v_i)\) is monotone.
  - e.g. if \(D\) is uniform, \(\phi(v_i) = v_i - (1 - v_i) = 2v_i - 1\)
  - Note that \(\phi^{-1}(0)\) recovers the optimal \(p = 1/2\) for a single bidder.
What do revenue maximizing auctions look like? (when $v_i$ drawn iid from regular $D$)

- We give the item to bidder $i^* = \arg \max_i \phi(v_i)$ when $\phi(v_{i^*}) \geq 0$.

- Because $\phi$ is monotone, $i^* = \arg \max_i v_i$: the item goes to the highest bidder when $\phi(v_{i^*}) \geq 0$.

- Winner pays $v_{i^*} - \int v_{i^*} p^*_1 = p^*_{\phi}$, where:

  - $p^* = \max(max_i \neq i^* v_i, \phi - 1(0))$.

- i.e. it’s just a Vickrey auction with a reserve price of $\phi - 1(0)$!

Remarkable — Simple eBay style auction is the best possible.
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  ▶ Each bidder has their own virtual valuation function $\phi_i(v_i)$.
  ▶ Auction no longer so natural. e.g. high bidder no longer necessarily wins.
▶ Doesn’t extend beyond single parameter domains...
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- The Vickrey auction yields no revenue selling to a single bidder, whereas when $D$ is uniform over $[0, 1]$ we can get expected revenue $1/4$. 

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- What about a Vickrey auction with 2 bidders?

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The Bulow/Klemperer Theorem

**Theorem**

Consider bidders drawn i.i.d. from a regular distribution $D$. For any $n \geq 1$, the Vickrey auction with $n + 1$ bidders has higher expected revenue than the revenue optimal auction with $n$ bidders.
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Consider bidders drawn i.i.d. from a regular distribution $D$. For any $n \geq 1$, the Vickrey auction with $n + 1$ bidders has higher expected revenue than the revenue optimal auction with $n$ bidders. So recruiting just one extra bidder is worth more than optimizing revenue for the current population.
The Bulow/Klemperer Theorem

Consider the hypothetical auction $A$ for $n + 1$ bidders:

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Observations:

1. The revenue of $A$ is exactly equal to the optimal revenue obtainable from $n$ bidders.
2. $A$ always allocates the item.
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- Recall that $E_v[\sum_i P_i(v)] = E[\sum_i \phi_i(v_i) \cdot X_i(v)]$. 
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- **Claim**: The Vickrey mechanism is obtained the maximum revenue amongst all mechanisms that always allocate the item.
- Recall that $E_v[\sum_i P_i(v)] = E[\sum_i \phi_i(v_i) \cdot X_i(v)]$.
- We can maximize the RHS (subject to always allocating the item) by always allocating to $\text{arg max}_i \phi(v_i)$.
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- We can maximize the RHS (subject to always allocating the item) by always allocating to $\text{arg max}_i \phi(v_i)$.
- Since $D$ is regular, $\phi$ is monotone: this is $\text{arg max}_i v_i$ — the Vickrey allocation!
But...

- **Claim**: The Vickrey mechanism is obtains the maximum revenue amongst all mechanisms that always allocate the item.
- Recall that $E_v[\sum_i P_i(v)] = E[\sum_i \phi_i(v) \cdot X_i(v)]$.
- We can maximize the RHS (subject to always allocating the item) by always allocating to $\arg \max_i \phi(v_i)$.
- Since $D$ is regular, $\phi$ is monotone: this is $\arg \max_i v_i$ — the Vickrey allocation!
- So: The Vickrey-auction with $n+1$ bidders has only higher revenue than the optimal $n$ bidder auction.
Thanks!

See you next class — stay healthy, and wear a mask!