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- And give a formalization of Adam Smith’s “Invisible Hand”
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The thesis (in our simple model): simple, decentralized market dynamics lead to efficient outcomes.
A Model

Suppose we have:

1. $m$ goods $G$ for sale
2. $n$ buyers $i$ who each have valuation functions over bundles, $v_i: 2^G \rightarrow [0, 1]$.

Buyers have quasi-linear utility functions: If each good $j \in G$ has a price $p_j$, then a buyer $i$ gets utility for buying a bundle $S \subseteq G$:

$$u_i(S) = v_i(S) - \sum_{j \in S} p_j$$

Questions:

How should we price and allocate goods so that everyone is happy with their allocation? Is this even possible? If it is, can we do so and also achieve a high welfare allocation?
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**Questions:** How we should price and allocate goods so that everyone is happy with their allocation. Is this even possible? If it is, can we do so and also achieve a high welfare allocation?
Some Definitions

First, feasibility:

**Definition**
An allocation $S_1, \ldots, S_n \subseteq G$ is *feasible* if for all $i \neq j$, $S_i \cap S_j = \emptyset$

We write $\text{OPT}$ to denote the socially optimal feasible allocation:

$$\text{OPT} = \max_{S_1, \ldots, S_n \text{ feasible}} \sum_i v_i(S)$$
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What is the right notion of equilibrium in a market?
Some Definitions

Definition
A set of prices $p$ together with an allocation $S_1, \ldots, S_n$ form an ($\epsilon$-approximate) *Walrasian equilibrium* if:

1. $S_1, \ldots, S_n$ is feasible, and
2. For all $i$, buyer $i$ is receiving his ($\epsilon$) most preferred bundle given the prices:

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq \max_{S^* \subseteq G} \left( v_i(S^*) - \sum_{j \in S^*} p_j \right) - \epsilon$$

and,

3. All unallocated items have zero price: for all $j \not\in S_1 \cup \ldots \cup S_n$, $p_j = 0$. 
Walrasian Equilibrium

At Walrasian equilibrium, no buyer wants to buy a different bundle, and the seller does not want to lower any of the prices – the only things that aren’t selling can’t sell (they already have price 0).
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1. Do Walrasian equilibria always exist?
Walrasian Equilibrium

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Some Questions:

1. Do Walrasian equilibria always exist?
2. If so, are they compatible with social welfare maximization?
The 2nd Question 1st

Theorem

If $S_1, \ldots, S_n$ form an $\epsilon$-Walrasian equilibrium allocation, then they achieve nearly optimal welfare. In particular:

$$\sum_i v_i(S_i) \geq OPT - \epsilon n$$
Proof

1. Let $p$ be the corresponding Walrasian equilibrium prices, and let $S'_1, \ldots, S'_n$ be any other feasible allocation.
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2. We know from the 2nd Walrasian equilibrium condition that for every player $i$, we have:

\[ v_i(S_i) - \sum_{j \in S_i} p_j \geq v_i(S'_i) - \sum_{j \in S'_i} p_j - \epsilon \]
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3. Summing over buyers:

$$\sum_i \left( v_i(S_i) - \sum_{j \in S_i} p_j \right) \geq \sum_i \left( v_i(S'_i) - \sum_{j \in S'_i} p_j \right) - \epsilon n$$
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4. Reordering:

\[
\sum_i v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_i v_i(S'_i) - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j - \epsilon n
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\sum_{i} v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_{i} v_i(S'_i) - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j - \epsilon n
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2. Rewriting:

\[ \sum_{i} v_i(S_i) \geq \sum_{i} v_i(S'_i) + (\sum_{j} p_j - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j) - \epsilon n \geq \sum_{i} v_i(S'_i) - \epsilon n \]
Proof

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\sum_i v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_i v_i(S'_i) - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j - \epsilon n
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\]

3. Finally, taking \( S'_1, \ldots, S'_n \) to be the optimal allocation gives the theorem. (Tada!)
1. We’ll start with a simple case: unit demand buyers (want to buy only 1 item):

   \[ v_i(S) = \max_{j \in S} v_i(\{j\}) \]

   We can think about such a valuation function as being determined by just \( m \) numbers, one for each good:
   
   \[ v_{i,j} \equiv v_i(\{j\}) \leq 1 \]
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**Theorem**

*For any set of unit demand buyers, a Walrasian equilibrium always exists.*
Proof

1. We’ll give a constructive proof: An algorithm for finding a Walrasian equilibrium.
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6. Deferred acceptance like...
Proof

Algorithm 1 The Ascending Price Auction with increment $\epsilon$.

For all $j \in G$, set $p_j = 0$, $\mu(j) = \emptyset$.

while There exist any unmatched bidders do

for Each unmatched bidder $i$ do

$i$ “bids” on $j^* = \arg \max_j (v_{i,j} - p_j)$ if $v_{i,j^*} - p_{j^*} > 0$. Otherwise, bidder $i$ drops out. (and is “matched” to nothing): $\mu(j^*)$ is now unmatched. Set $\mu(j^*) \leftarrow i$

$p_{j^*} \leftarrow p_{j^*} + \epsilon$

end for

end while

Output $(p, \mu)$. 

Proof

Lemma

The ascending price auction halts after at most \( \frac{n}{\epsilon} \) bids.

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1. Claim: At any point during the algorithm, we must have:
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3. For any fixed good $j$, $p_j \leq 1$. (no bidder bids on any good $j$ such that $v_{i,j} - p_j < 0$, and $v_{i,j} \leq 1$ for all $i, j$.)
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6. (Lemma Tada!)
Proof

Lemma
The output \((p, \mu)\) of the ascending price auction is an \(\epsilon\)-approximate Walrasian equilibrium.
Proof

We’ll verify the 3 conditions:

1. By construction it outputs a feasible allocation.
2. If good \( j \) is unallocated, it must never have received a bid in the auction, and hence \( p_j = 0 \).
3. Finally: \[ v_i - \mu(i) - \mu(i) \geq \max_j (v_i, j - p_j) - \epsilon. \] This is because...
4. at the time bidder \( i \) was matched to good \( \mu(i) \), we must have had: \[ \mu(i) \in \arg \max_j (v_i, j - p_j) \]
5. Since that time \( p_j \) increased by \( \epsilon \), no other price has decreased.
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1. We will see on the homework that Walrasian equilibrium need not exist for all valuation functions.
Beyond Unit Demand Valuations

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\[
\begin{align*}
\text{For each unsatisfied bidder } i: \\
&\text{bids on every item she is not the high bidder on in a set } S^* \in \arg \max_{S \subseteq G} (v_i(S) - \Sigma_{j \in S} p_j) \\
&\forall j \in S^*, \mu(j) \leftarrow i, p_j \leftarrow p_j + \epsilon/m.
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4.2 For all $j \in S^*$, $\mu(j) \leftarrow i$, $p_j \leftarrow p_j + \epsilon/m$. 
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4. We can formalize this.
Beyond Unit Demand Valuations

1. For price vectors $p, p'$, write $p \preceq p'$ to mean that $p_j \leq p'_j$ for all $j$. Let $w_i(p) = \arg \max_{S \subseteq G} (v_i(S) - \sum_{j \in S} p_j)$ be player $i$’s demand set at prices $p$.

Definition
Valuation function $v_i$ satisfies the gross substitutes property if for every $p \preceq p'$ and for every $S \in w_i(p)$, if $S' = \{j \in S : p_j = p'_j\}$, then there exits $S^* \in w_i(p')$ such that $S' \subseteq S^*$. In other words, “Raising the prices on goods $j \neq i$ doesn’t decrease a bidder’s demand for good $j$”.

2. This is what we need: Any good for which bidder $i$ has not been out-bid on has not had its price raised, and so must still be part of a bundle in bidder $i$’s demand set.

3. Hence, we have:
Theorem
In any market in which all buyers satisfy the gross substitutes condition, Walrasian equilibria exist.
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Valuation function \( v_i \) satisfies the **gross substitutes** property if for every \( p \preceq p' \) and for every \( S \in w_i(p) \), if \( S' = \{ j \in S : p_j = p'_j \} \), then there exits \( S^* \in w_i(p') \) such that \( S' \subseteq S^* \).

In other words, “Raising the prices on goods \( j \neq i \) doesn’t decrease a bidder’s demand for good \( j \”).

2. This is what we need: Any good for which bidder \( i \) has not been out-bid on has not had its price raised, and so must still be part of a bundle in bidder \( i \)'s demand set.

3. Hence, we have:

**Theorem**

*In any market in which all buyers satisfy the gross substitutes condition, Walrasian equilibria exist.*
Thanks!

See you next class — stay healthy, and wear a mask!