Congestion Games

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Overview

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- We'll study a simple, natural dynamic, and show it converges to Nash equilibrium.

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- We'll study a simple, natural dynamic, and show it converges to Nash equilibrium.
- Our first "computationally plausible" set of predictions in a large interaction.

Large Games

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A: We need k^n numbers just to encode a single utility function.

Unreasonable to expect anyone to understand such an object.
So: we need to think about structured, concisely defined games.

Example 1: Traffic Routing

Congestion Games

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4. For each facility $j \in F$, a cost function $\ell_j : \{0, \dots, n\} \to \mathbb{R}_{\geq 0}$. $\ell_j(k)$ represents "the cost of facility j when k players are using it".

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Player costs are then defined as follows. For action profile $a = (a_1, \ldots, a_n)$ define $n_j(a) = |\{i : j \in a_i\}|$ to be the number of players using facility j. Then the cost of agent i is:

$$c_i(a) = \sum_{j \in a_i} \ell_j(n_j(a))$$

Example 2: Network Creation

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- i.e. are pure strategy Nash equilibria computationally plausible predictions?
- Lets study a simple dynamic...

Best (Better) Response Dynamics

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1. Players start playing *arbitrary* actions.
2. In arbitrary order, players take turns changing their action if doing so can improve their utility.
3. Forever...

Best (Better) Response Dynamics

Algorithm 1 Best Response Dynamics

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i such that $a_i \notin \arg\min_{a \in A_i} c_i(a, a_{-i})$ **do**
  Let a'_i be such that $c_i(a'_i, a_{-i}) < c(a)$.
  Set $a_i = a'_i$.
end while
Halt and return a.

Best (Better) Response Dynamics

Algorithm 2 Best Response Dynamics

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i such that $a_i \notin \arg\min_{a \in A_i} c_i(a, a_{-i})$ **do**
  Let a_i' be such that $c_i(a_i', a_{-i}) < c(a)$.
  Set $a_i = a_i'$.
end while
Halt and return a.

Claim

If best response dynamics halts, it returns a pure strategy Nash equilibrium.

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Algorithm 3 Best Response Dynamics

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Halt and return a.

Claim

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Proof.

Immediate from halting condition – by definition, every player must be playing a best response. □

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Best response dynamics always halt in congestion games.

Corollary

All congestion games have at least one pure strategy Nash equilibrium.

Analysis of BRD in Congestion Games

1. Consider the *potential function* $\phi : A \to \mathbb{R}$:

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

(Note: *not* social welfare)

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2. How does ϕ change in one round of BRD? Say i switches from a_i to $b_i \in A_i$.
3. Well... We know it must have decreased player i's cost:

$$\begin{aligned} \Delta c_i &\equiv c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &< 0 \end{aligned}$$

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2. Therefore, the change in potential is strictly *negative*
3. So... since ϕ can take on only finitely many values, this cannot go on forever.
4. And hence BRD halts in congestion games...
5. Which proves the *existence* of pure strategy Nash equilibria!

Efficiency

But... How long does it take?

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Our proof gives only an exponential convergence bound... And it might really take that long!

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Lets consider approximation...

Approximation

Definition

An action profile $a \in A$ is an ϵ-approximate pure strategy Nash equilibrium if for every player i, and for every action $a_i' \in A_i$:

$$c_i(a_i, a_{-i}) \leq c_i(a_i', a_{-i}) + \epsilon$$

i.e. nobody can gain more than ϵ by deviating.

.

Approximate Best Response Dynamics

Algorithm 4 FindApproxNash(ϵ)

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i, a'_i such that $c_i(a'_i, a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon$ **do**
  Set $a_i = \arg\min_{a \in A_i} c_i(a, a_{-i})$
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Claim

If FindApproxNash(ϵ) halts, it returns an ϵ-approximate pure strategy Nash equilibrium

Approximate Best Response Dynamics

Algorithm 6 FindApproxNash(ϵ)

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Proof.

Immediately, by definition. □

Analysis

Theorem

In any congestion game, FindApproxNash(ϵ) halts after at most:

$$\frac{n \cdot m \cdot c_{max}}{\epsilon}$$

steps, where $c_{max} = \max_{j,k} \ell_j(k)$ is the maximum facility cost.

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We revisit the potential function ϕ. Recall that $\Delta c_i = \Delta\phi$ on any round when player i moves.

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Observe also that at every round, $\phi \geq 0$, and

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} c_j(k) \leq n \cdot m \cdot c_{max}$$

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By definition of the algorithm, we have $\Delta c_i = \Delta\phi \leq -\epsilon$ at every round, and so the theorem follows. □

Thanks!

See you next class — stay healthy, and wear a mask!