Congestion Games

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Overview

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- We’ll study a simple, natural dynamic, and show it converges to Nash equilibrium.
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We’ll study a simple, natural dynamic, and show it converges to Nash equilibrium.
Our first “computationally plausible” set of predictions in a large interaction.
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A: We need $k^n$ numbers just to encode a single utility function.
Large Games

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Unreasonable to expect anyone to understand such an object.
Large Games

**Q:** How many numbers do we need to write down to represent an \( n \) player \( k \) action game?

**A:** We need \( k^n \) numbers just to encode a single utility function.

Unreasonable to expect anyone to understand such an object. So: we need to think about structured, concisely defined games.
Example 1: Traffic Routing
Congestion Games

Convention: Players have cost functions they want to minimize rather than utility functions they want to maximize.
Congestion Games

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Definition
A congestion game is defined by:

1. A set of *n* players *P*
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1. A set of $n$ players $P$
2. A set of $m$ facilities $F$
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Definition
A congestion game is defined by:

1. A set of $n$ players $P$
2. A set of $m$ facilities $F$
3. For each player $i$, a set of actions $A_i$. Each action $a_i \in A_i$ represents a subset of the facilities: $a_i \subseteq F$.

Player costs are then defined as follows. For action profile $a = (a_1, \ldots, a_n)$ define $n_j(a) = |\{i : j \in a_i\}|$ to be the number of players using facility $j$. Then the cost of agent $i$ is:

$$c_i(a) = \sum_{j \in a_i} \ell_j(n_j(a))$$

i.e. the total cost of the facilities she is using.
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4. For each facility $j \in F$, a cost function $\ell_j : \{0, \ldots, n\} \rightarrow \mathbb{R}_{\geq 0}$. $\ell_j(k)$ represents “the cost of facility $j$ when $k$ players are using it”.

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Example 2: Network Creation
Ok, so we can concisely describe a large game...
Congestion games

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- So what? What can we do with it?
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- Can computationally bounded, uncoordinated players find one?
- i.e. are pure strategy Nash equilibria computationally plausible predictions?
- Lets study a simple dynamic...
Best (Better) Response Dynamics

The basic idea:
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1. Players start playing *arbitrary* actions.
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1. Players start playing *arbitrary* actions.
2. In arbitrary order, players take turns changing their action if doing so can improve their utility.
3. Forever...
Algorithm 1 Best Response Dynamics

Initialize \( a = (a_1, \ldots, a_n) \) to be an arbitrary action profile.

while There exists \( i \) such that \( a_i \not\in \arg\min_{a \in A_i} c_i(a, a_{-i}) \) do
  Let \( a'_i \) be such that \( c_i(a'_i, a_{-i}) < c(a) \).
  Set \( a_i = a'_i \).
end while

Halt and return \( a \).
Best (Better) Response Dynamics

Algorithm 2 Best Response Dynamics

Initialize \( a = (a_1, \ldots, a_n) \) to be an arbitrary action profile.

while There exists \( i \) such that \( a_i \not\in \arg \min_{a \in A_i} c_i(a, a_{-i}) \) do

Let \( a_i' \) be such that \( c_i(a_i', a_{-i}) < c(a) \).

Set \( a_i = a_i' \).

end while

Halt and return \( a \).

Claim

If best response dynamics halts, it returns a pure strategy Nash equilibrium.
Algorithm 3 Best Response Dynamics

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.

while There exists $i$ such that $a_i \notin \arg \min_{a \in A_i} c_i(a, a_{-i})$ do

    Let $a'_i$ be such that $c_i(a'_i, a_{-i}) < c(a)$.
    Set $a_i = a'_i$.

end while

Halt and return $a$.

Claim

If best response dynamics halts, it returns a pure strategy Nash equilibrium.

Proof.

Immediate from halting condition – by definition, every player must be playing a best response. □
Best (Better) Response Dynamics

Does best response dynamics always halt?
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No: Consider matching pennies/Rock Paper Scissors.
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Theorem

*Best response dynamics always halt in congestion games.*
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Theorem
*Best response dynamics always halt in congestion games.*

Corollary
*All congestion games have at least one pure strategy Nash equilibrium.*
Analysis of BRD in Congestion Games

1. Consider the potential function $\phi : A \to \mathbb{R}$:

$$
\phi(a) = \sum_{j=1}^{m} \left( \sum_{k=1}^{n_j(a)} \ell_j(k) \right)
$$

(Note: not social welfare)
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2. How does $\phi$ change in one round of BRD? Say $i$ switches from $a_i$ to $b_i \in A_i$.

3. Well... We know it must have decreased player $i$'s cost:

$$\Delta c_i \equiv c_i(b_i, a_{-i}) - c_i(a_i, a_{-i})$$

$$= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s))$$

$$< 0$$
Analysis of BRD in Congestion Games

\[ \phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k) \]

1. The change in potential is:
   \[ \Delta \phi \equiv \phi(b_i, a^{-i}) - \phi(a_i, a^{-i}) = \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \]

2. Therefore, the change in potential is strictly negative.

3. So... since \( \phi \) can take on only finitely many values, this cannot go on forever.

4. And hence BRD halts in congestion games...

5. Which proves the existence of pure strategy Nash equilibria!
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2. Therefore, the change in potential is strictly negative.

3. So... since \( \phi \) can take on only finitely many values, this cannot go on forever.

4. And hence BRD halts in congestion games...

5. Which proves the existence of pure strategy Nash equilibria!
But... How long does it take?
Efficiency

But... How long does it take?
Our proof gives only an exponential convergence bound... And it might really take that long!
Efficiency

But... How long does it take?
Our proof gives only an exponential convergence bound... And it might really take that long!
Lets consider approximation...
Approximation

Definition
An action profile \( a \in A \) is an \( \epsilon \)-approximate pure strategy Nash equilibrium if for every player \( i \), and for every action \( a_i' \in A_i \):

\[
c_i(a_i, a_{-i}) \leq c_i(a_i', a_{-i}) + \epsilon
\]

i.e. nobody can gain more than \( \epsilon \) by deviating.
Approximate Best Response Dynamics

\textbf{Algorithm 4 FindApproxNash}(\epsilon)

\begin{align*}
\text{Initialize } a &= (a_1, \ldots, a_n) \text{ to be an arbitrary action profile.} \\
\text{while } &\text{There exists } i, a_i' \text{ such that } c_i(a_i', a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon \text{ do} \\
&\text{Set } a_i = \arg\min_{a \in A_i} c_i(a, a_{-i}) \\
\text{end while} \\
\text{Halt} \text{ and return } a.
\end{align*}
Approximate Best Response Dynamics

**Algorithm 5** FindApproxNash(\(\epsilon\))

- **Initialize** \(a = (a_1, \ldots, a_n)\) to be an arbitrary action profile.
- **while** There exists \(i, a'_i\) such that \(c_i(a'_i, a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon\) do
  - Set \(a_i = \arg \min_{a \in A_i} c_i(a, a_{-i})\)
- **end while**
- **Halt** and return \(a\).

**Claim**

*If FindApproxNash(\(\epsilon\)) halts, it returns an \(\epsilon\)-approximate pure strategy Nash equilibrium*
Approximate Best Response Dynamics

Algorithm 6 FindApproxNash(\(\epsilon\))

Initialize \(a = (a_1, \ldots, a_n)\) to be an arbitrary action profile.

while There exists \(i, a'_i\) such that \(c_i(a'_i, a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon\) do

Set \(a_i = \arg \min_{a \in A_i} c_i(a, a_{-i})\)

end while

Halt and return \(a\).

Claim

If FindApproxNash(\(\epsilon\)) halts, it returns an \(\epsilon\)-approximate pure strategy Nash equilibrium

Proof.

Immediately, by definition.
Analysis

Theorem

In any congestion game, \textit{FindApproxNash}(\epsilon) halts after at most:

\[
\frac{n \cdot m \cdot c_{\text{max}}}{\epsilon}
\]

steps, where \( c_{\text{max}} = \max_{j,k} \ell_j(k) \) is the maximum facility cost.
Analysis

Theorem

In any congestion game, $\text{FindApproxNash}(\epsilon)$ halts after at most:

$$\frac{n \cdot m \cdot c_{\text{max}}}{\epsilon}$$

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Proof.

We revisit the potential function $\phi$. Recall that $\Delta c_i = \Delta \phi$ on any round when player $i$ moves.
Analysis

Theorem
In any congestion game, $\text{FindApproxNash}(\epsilon)$ halts after at most:

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steps, where $c_{\text{max}} = \max_{j,k} \ell_j(k)$ is the maximum facility cost.

Proof.
We revisit the potential function $\phi$. Recall that $\Delta c_i = \Delta \phi$ on any round when player $i$ moves.
Observe also that at every round, $\phi \geq 0$, and

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} c_j(k) \leq n \cdot m \cdot c_{\text{max}}$$
Analysis

Theorem

*In any congestion game, FindApproxNash(\(\epsilon\)) halts after at most:*

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*steps, where \(c_{\text{max}} = \max_{j,k} \ell_j(k)\) is the maximum facility cost.*

Proof.

We revisit the potential function \(\phi\). Recall that \(\Delta c_i = \Delta \phi\) on any round when player \(i\) moves.

Observe also that at every round, \(\phi \geq 0\), and

\[
\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} c_j(k) \leq n \cdot m \cdot c_{\text{max}}
\]

By definition of the algorithm, we have \(\Delta c_i = \Delta \phi \leq -\epsilon\) at every round, and so the theorem follows.
Thanks!

See you next class — stay healthy, and wear a mask!