Basic Definitions

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Overview

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- Games, Best Responses, Dominant Strategies, Iterated Elimination...

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- Solution concepts: Nash equilibrium

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- A utility function $u_i : A \to \mathbb{R}$ for each player $i \in P$.

Utility Maximization

Basic assumption: players will always try and act so as to maximize their utility.

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Definition

The *best-response* to a set of actions $a_{-i} \in A_{-i}$ for a player i is any action $a_i \in A_i$ that maximizes $u_i(a_i, a_{-i})$:

$$a_i \in \arg\max_{a \in A_i} u_i(a, a_{-i})$$

Interlude

Question: Is game theory just for sociopaths?

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Answer: Not necessarily. (Assumes only that people have consistent preferences)

The General Idea for Prediction

"In any stable situation, all players should be playing a best response."

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"In any stable situation, all players should be playing a best response."

(Otherwise, by definition, the situation would not be stable – somebody would want to change their action.)

When are there stable solutions?

Definition

For a player i, an action $a \in A_i$ (weakly) dominates action $a' \in A_i$ if it is always beneficial to play a over a'. That is, if for all $a_{-i} \in A_{-i}$:

$$u_i(a, a_{-i}) \geq u_i(a', a_{-i})$$

and the inequality is strict for some $a_{-i} \in A_{-i}$.

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and the inequality is strict for some $a_{-i} \in A_{-i}$.

Can normally eliminate dominated strategies from consideration – there is never a situation in which they are the (unique) best response.

Dominant Strategies

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An action $a \in A_i$ is *dominant* for player i if it weakly dominates all actions $a' \neq a \in A_i$.

1. A very strong guarantee – Always a best response.
2. No need to reason about what your opponents are doing.

Dominant Strategy Equilibrium

Dominant strategies normally don't exist, but when they do, predictions are easy.

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Definition

An action profile $a = (a_1, \ldots, a_n) \in A$ is a *dominant strategy equilibrium* of the game $(P, \{A_i\}, \{u_i\})$ if for every $i \in P$, a_i is a dominant strategy for player i.

Example: Prisoner's Dilemma

	Confess	Silent
Confess	$(1, 1)$	$(5, 0)$
Silent	$(0, 5)$	$(3, 3)$

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(Confess, Confess) is a dominant strategy equilibrium is Prisoner's Dilemma.

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- It still makes sense to eliminate *dominated* strategies from consideration.
- Sometimes, once you've done this, new strategies have become dominated.
- We can consider eliminating dominated strategies *iteratively*.
- If we are lucky, "iterated elimination of dominated strategies" leads to a unique surviving strategy profile.

Iterated Elimination: Example 1

	X	Y
A	$(5, 2)$	$(4, 2)$
B	$(3, 1)$	$(3, 2)$
C	$(2, 1)$	$(4, 1)$
D	$(4, 3)$	$(5, 4)$

Figure: Example 1

Iterated Elimination: Example 2

	V	W	X	Y	Z
A	$(4,-1)$	$(3,0)$	$(-3,1)$	$(-1,4)$	$(-2,0)$
B	$(-1,1)$	$(2,2)$	$(2,3)$	$(-1,0)$	$(2,5)$
C	$(2,1)$	$(-1,-1)$	$(0,4)$	$(4,-1)$	$(0,2)$
D	$(1,6)$	$(-3,0)$	$(-1,4)$	$(1,1)$	$(-1,4)$
E	$(0,0)$	$(1,4)$	$(-3,1)$	$(-2,3)$	$(-1,-1)$

Figure: Example 2

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Definition

A profile of actions $a = (a_1, \ldots, a_n) \in A$ is a *pure strategy Nash Equilibrium* if for each player $i \in P$ and for all $a_i' \in A_i$:

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$$

i.e. simultaneously, all players are playing a best response to one another.

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Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

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Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

Proof.

Homework! □

Problem 1: They don't always exist.

	Heads	Tails
Heads	$(1, -1)$	$(-1, 1)$
Tails	$(-1, 1)$	$(1, -1)$

Figure: Matching Pennies

Problem 2: They aren't always unique.

	Bach	Stravinsky
Bach	$(5, 1)$	$(0, 0)$
Stravinsky	$(0, 0)$	$(1, 5)$

Figure: Bach of Stravinsky

Question: What to Predict when No Pure Nash Equilibria?

Definition

A two-player game is *zero-sum* if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of of both players sum to zero at every action profile)

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Definition

A *mixed-strategy* $p_i \in \Delta A_i$ is a probability distribution over actions $a_i \in A_i$: i.e. a set of numbers $p_i(a_i)$ such that:

1. $p_i(a_i) \geq 0$ for all $a_i \in A_i$
2. $\sum_{a_i \in A_i} p_i(a_i) = 1$.

For $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$, we write:

$$u_i(p) = E_{a_i \sim p_i}[u_i(a)]$$

Mixed Strategy Nash Equilibria

Definition

A *mixed strategy Nash equilibrium* is a tuple $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$ such that for all i, and for all $a_i \in A_i$:

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Theorem (Nash)

Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.

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Theorem (Nash)

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But... The proof is non-constructive, so its not necessarily clear how to find one of these, even though they exist

Thanks!

See you next class — stay healthy, and wear a mask!