Algorithmic Game Theory: Problem Set 6

Due on Tuesday, April 23

Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment.

Posted Pricings for Multi-Unit Prophets (15 points)

A natural extension of our posted pricings model in lecture is to allow the seller to sell $k \geq 1$ units of the same good to buyers (who each can buy up to 1 good). We'll explore a way to asymptotically approach optimal welfare for large k. $\forall i \in [1 \ldots n]$, let $v_i \sim \mathcal{D}_i$ be the random variable denoting the buyer's valuation of the good in round i. Let OPT be the sum of the k largest v_is, and let $\mathbf{1}_i^t(v_i)$ be the indicator random variable for whether the i^{th} buyer with valuation v_i can purchase the good at threshold price t:

$$\mathbf{1}_i^t(v_i) = \begin{cases} 1 & v_i \geq t \\ 0 & v_i < t \end{cases}$$

We'd like to set a threshold t so the seller is likely to sell most of the k items without overselling. With this in mind, for a fixed set of realized v_is, let t satisfy the following:

$$\mathbb{E}[S] := \sum_{i=1}^{n} \mathbb{E}[\mathbf{1}_i^t(v_i)] = k - \delta$$

We'll define δ later. In essence, δ ensures that with high probability, no more than k items are demanded. To prove this, we'll need the following:

Theorem: Multiplicative Chernoff Bound

$\forall i \in [1 \ldots n]$, let $X_i \sim \mathcal{D}_i$ be independent draws, where $X_i \in [0, 1]$. Let $X = \sum_{i=1}^{n} X_i$. Then, we have the following inequality for $\alpha \in [0, 1]$:

$$\Pr\Big[|X - \mathbb{E}[X]| \geq \alpha \cdot \mathbb{E}[X]\Big] \leq 2\exp\big(-\alpha^2 \cdot \mathbb{E}[X]/3\big)$$

We would like to ensure that with high probability, $|S - \mathbb{E}[S]| \leq \delta \implies S \in [k - 2\delta, k]$. I.e., it is unlikely that with threshold t, there are $< k - 2\delta$ or $> k$ buyers that can purchase the good.

1. (5 pts) Suppose that $S \in [k - 2\delta, k]$. Prove that $\mathbb{E}[\text{Welfare}] \geq (1 - \frac{2\delta}{k})\,\text{OPT}$.
2. (5 pts) Prove that when $\delta = \sqrt{c \cdot k \log k}$ for some constant c, $S \in [k - 2\delta, k]$ with probability $1 - O(\frac{1}{k})$.
3. (5 pts) Prove that $\mathbb{E}[\text{Welfare}] = \left(1 - O\left(\sqrt{\frac{\log k}{k}}\right)\right)\text{OPT}$, which shows that this approach yields a $\left(1 + O\left(\sqrt{\frac{\log k}{k}}\right)\right)$-approximation to optimal welfare.

Knapsack Auctions (15 pts)

1. In class, we showed a 2-approximation algorithm for the knapsack auction, and proved that it was a monotone allocation rule. Give an explicit description of the payment rule which makes it a dominant strategy to report valuations truthfully. (5 pts)

2. Now consider a variant of the knapsack auction in which each bidder i has to report *both* his value v_i *and* his size w_i. This is no longer a single parameter domain. An allocation rule $x(v', w')$ now specifies the amount of capacity allocated to each bidder as a function of their reported bids and sizes. The utility of buyer i is defined to be $v_i - p_i(v', w')$ if she gets her full capacity – i.e. if $x_i(v', w') \geq w_i$, and to be $-p_i(v, w)$ otherwise (i.e. she gets no value for getting capacity less than her full size). Consider the 2-approximation we considered in class that simply takes the reported sizes w' at face value (and define $x_i(v', w') = w'_i$ if the algorithm allocates buyer i with these reports, and $x_i(v', w') = 0$ otherwise), and uses the same payment rule computed above. Does this mechanism make reporting true values and sizes a dominant strategy? Prove it if so, or give an explicit counter-example. (10 pts)

Uniqueness of the Groves Mechanism (20 pts)

From lecture, we know that the Groves mechanism is efficient and DSIC. We've also seen that the VCG mechanism, which instantiates the Groves mechanism, achieves multiple desiderata but has certain flaws (e.g., may be computationally inefficient). Could there be efficient, DSIC mechanisms besides the Groves mechanism that rectify these flaws? Unfortunately, this is not the case.

1. (5 pts) Consider a mechanism (X, P) which is efficient (i.e., $X(v) = \arg\max_{a \in A} \sum_i v_i(a)$). Prove (X, P) is an instantiation of the Groves mechanism if and only if $\forall v_i \in V$ with $v'_i \neq v_i$, $\forall v_{-i} \in V^{n-1}$:

$$P(v_i, v_{-i})_i - P(v'_i, v_{-i})_i = \sum_{j \neq i} v_j\big(X(v'_i, v_{-i})\big) - \sum_{j \neq i} v_j\big(X(v_i, v_{-i})\big)$$

2. (5 pts) Now, suppose that (X, P) is also DSIC. Prove that if agent i's report does not change the output of X, agent i's payment does not change. I.e., $\forall i \in [1 \dots n]$, $\forall v_i, v'_i \in V$, $\forall v_{-i} \in V^{n-1}$:

$$X(v_i, v_{-i}) = X(v'_i, v_{-i}) \implies P(v_i, v_{-i})_i = P(v'_i, v_{-i})_i$$

3. (10 pts) Use parts a). and b). to prove that any mechanism (X, P) that is efficient and DSIC is an instantiation of the Groves mechanism. For ease, assume that the set of alternatives A is finite.