Algorithmic Game Theory: Problem Set 5

Due on Tuesday, April 9

Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment.

Walrasian Equilibrium (15 pts)

1. We saw in lecture 14 that when buyers have unit demand valuations (or more generally, *gross substitutes* valuations) Walrasian equilibria always exist. In this problem you will give a simple example with two goods and two buyers showing that for general valuation functions, they need not exist. Specifically, consider a market consisting of two goods: $\mathcal{G} = \{PB, J\}$. Describe two valuation functions $v_1 : 2^{\mathcal{G}} \to \mathbb{R}_{\geq 0}, v_2 : 2^{\mathcal{G}} \to \mathbb{R}_{\geq 0}$ such that Walrasian equilibria do not exist in the market with two buyers with valuations v_1 and v_2. *Prove* that no Walrasian equilibrium exists in your example. (5 pts).

2. Walrasian equilibria need not be unique. Consider a market in which there are multiple Walrasian equilibria, with two such equilibria denoted as price/allocation pairs $(p, (S_1, \ldots, S_n))$ and $(p', (S'_1, \ldots, S'_n))$ respectively. Prove that in this case, we can swap the price vectors while preserving the equilibrium property — i.e. prove that $(p, (S'_1, \ldots, S'_n))$ and $(p', (S_1, \ldots, S_n))$ are also Walrasian equilibria. (10 pts)

Budget Balance (20 pts)

We know that the VCG mechanism is no-deficit: $\sum_i P(v)_i \geq 0$ for all $v \in V^n$. However, there are scenarios in which we might like a *budget balanced* mechanism; i.e., $\sum_i P(v)_i = 0$. Here, each agent's payment is effectively redistributed to the other agents.

1. (10 pts) Prove that for $n \geq 2$, there exists a budget balanced Groves Mechanism $\iff$ for all $i \in [n]$, there exists a function $f_i : V^{n-1} \to \mathbb{R}$ such that for $a^* = X(v)$:

$$\sum_{i=1}^{n} v_i(a^*) = \sum_{i=1}^{n} f_i(v_{-i})$$

2. (10 pts) Derive a variant of the VCG mechanism that is always budget balanced, and is *approximately* social welfare maximizing. In particular, you should propose a mechanism (X, P), and prove that it has the following properties:

   (a) Dominant strategy incentive compatible

   (b) Individually Rational

(c) Budget balanced

(d) Approximately Social Welfare Maximizing:

$$\mathbb{E}[\sum_i v_i(X(v))] \geq \left(1 - \frac{1}{n}\right) \text{OPT}$$

Hint: Use the VCG mechanism in some way, and its ok to randomize.

Firms and Workers (10 points)

In this problem, we'll combine ideas from our study of Walrasian equilibrium and stable matching. Suppose there are n firms and m workers W, where the firms are interested in hiring workers. Each worker j has no cost for working and if paid a salary of $s_j \in [0, M]$ if hired by a firm has utility of s_j. Each firm i has a valuation function over sets of workers $S \subseteq W \to [0, mM]$. Assume that each v_i is additive over the workers; i.e.,

$$v_i(S) = \sum_{j \in S} v_i(\{j\}) \text{ for all } S \subseteq W$$

Firm i's utility function u_i for a set of workers S depends on v_i and each worker's salary:

$$u_i(S) = v_i(S) - \sum_{j \in S} s_j$$

Propose a bidding procedure for assigning workers to firms, and prove it has the following properties:

1. The procedure halts after at most O(mM) bids. If OPT denotes optimal social welfare for the firms, this procedure assigns workers S_i such that:

$$\sum_i v_i(S_i) \geq \text{OPT} - O(n)$$

2. There are no blocking pairs in the resulting assignment in the following sense: For every firm i and set of workers S, there *do not* exist different salaries $\{s'_j\}$ such that:

   (a) Firm i strictly prefers the workers S with salaries $\{s'_j\}$ to the procedure's outcome, and

   (b) Each worker $j \in S$ strictly prefers working at firm i with salary s'_j to the procedure's outcome, by at least a margin of 1. i.e. $s'_j \geq s_j + 1$.