Algorithmic Game Theory: Problem Set 4

Due online via GradeScope before the start of class on Tuesday, March 26

Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment. Ask questions on Slack.

University Housing (15 pts)

Many of you have likely gone through the laborious process of securing on-campus housing before. Setting housing population limits aside, it turns out that Penn's housing allocation mechanism isn't Pareto optimal anyhow. Let's explore a model involving upperclassmen and freshmen to see if we can improve on this.

Model:

- Let P_1 denote upperclassmen (i.e., previous tenants) and P_2 denote freshmen (i.e., new applicants).
- Let H_1 denote occupied rooms (by an upperclassman) and H_2 denote unoccupied rooms.
  - $|P_1| = |H_1|$, $|P_2| = |H_2|$
- Every student has a strict preference ordering over all rooms in $H_1 \cup H_2$.
- Upperclassmen have the option to return to their occupied room and not participate in the mechanism. We want it to be individually rational for the upperclassmen to participate in it.

Consider the following mechanism, analogous to top trading cycles in this model:

Algorithm 1 University Housing

$P^1 = P_1 \cup P_2$
$H^1 = H_1 \cup H_2$
$t = 1$
while $|P^t| > 0$ **do Construct** a graph $G^t = (V^t, E^t)$ where $V^t = P^t \cup H^t$ and E^t includes the following edges:
- Each remaining student points to their most preferred remaining room.
- Each occupied house with a remaining upperclassman tenant points to the tenant.
- All other houses point to the unmatched student with the lowest number. **Find** a cycle C^t and assign each player in it to the room they are pointing to.
  $P^{t+1} = P^t - \{\text{set of all players in } C^t\}$.
  $H^{t+1} = H^t - \{\text{set of all houses in } C^t\}$.
  $t = t + 1$
end while=0

1. (3 pts) Prove that this mechanism halts.

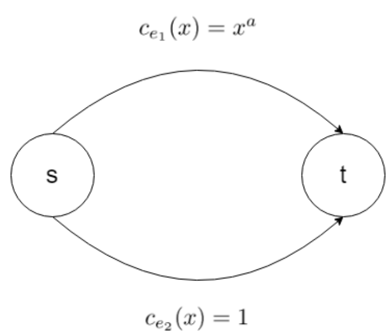
2. (3 pts) Prove that this mechanism produces a Pareto optimal allocation.
3. (4 pts) Prove that this mechanism is individually rational.
   Hint: This is particularly relevant for the upperclassmen.
4. (5 pts) Prove that this mechanism is dominant strategy incentive compatible.
   Hint: For some agent, let round T be the one in which they're assigned a room when reporting truthfully, and let round T' be the one when reporting dishonestly (in an arbitrary way). Consider cases when $T = T'$, $T > T'$, and $T < T'$, and show that in all cases, it is best to report truthfully.

Nonatomic Selfish Routing Games (20 points)

A routing game is described by a graph $G(V, E)$ that aims to model how traffic on a network like the internet operates. Each edge $e \in E$ has a cost function $c_e(x)$ associated with it. Players aim to minimize their individual cost as they traverse from the source node s to the sink node t. In a nonatomic routing game, players are modelled as a continuum, cumulatively controlling a unit flow in the graph, with each player controlling an infinitesimally small fraction of the flow. That is to say, the fraction of players that uses each edge is what contributes to cost; each individual player does not contribute much on their own — nothing in this limiting case.

In the particular nonatomic routing game we study in this problem, there are only 2 vertices and 2 edges.

- $V = \{s, t\}$, $E = \{e_1, e_2\}$
  - $e_1 = (s, t)$, $c_{e_1}(x) = x^a$ for some $a \geq 0$
  - $e_2 = (s, t)$, $c_{e_2}(x) = 1$



In particular, a "flow" (the continuous analogue of an action profile) in this simple game is specified by a single number $x \in [0, 1]$ which specifies what fraction of the population takes edge 1 — the remaining $1 - x$ fraction of the population takes edge 2. Since individuals are assumed to be infinitesimally small, their unilateral deviations would not affect the flow. Thus a flow is said to be at Nash equilibrium if all $s \to t$ paths have the same cost.

We define our objective function (i.e., social cost), which we want to minimize, as follows, where x is the fraction of players using e_1:

$$\text{Objective}(x) = x \cdot c_{e_1}(x) + (1 - x) \cdot c_{e_2}(1 - x)$$

Thus OPT (the minimal social cost) is:

$$= \min_{x \in [0,1]} x \cdot c_{e_1}(x) + (1 - x) \cdot c_{e_2}(1 - x)$$

1. (4 pts) Find a Nash equilibrium of this game (described by the fraction of players x that take the top edge), and prove that it is unique.

2. (2 pts) Calculate the social cost at this NE.

3. (4 pts) In order to minimize social cost, compute the optimal fraction x^* of players that must use e_1 in terms of a.

4. (5 pts) Compute Objective(x^*) in terms of α.

5. (3 pts) For $\alpha = 1$, what is the price of anarchy of this game? What is the price of stability?

6. (2 pts) Can you compute an upper bound for the price of anarchy in this class of games that holds uniformly for all α? If so, what is the upper bound, and if not, why?

School Choice Mechanisms (15 points)

School choice refers to a policy in which parents can choose which school their child will attend, as opposed to being assigned a public school based on where they reside. Since public schools have enrollment caps, mechanism design has been central to determine which students can enroll in different schools. In fact, mechanisms like deferred acceptance have been used for this exact purpose! But here we will consider a different school choice mechanism (not based on the deferred acceptance algorithm) and consider its properties.

Model:

- Let M and W denote sets of students and schools respectively. It is not necessarily true that $|M| = |W|$.
- Every student has a strict preference ordering over all schools, and vice-versa.
- Each student $m_i \in M$ will want 1 school.
- Each school $w_j \in W$ will want up to c_j students, for some $c_j \in \mathbb{Z}_+$.

Consider the following mechanism for school choice:

Algorithm 2 School Choice Mechanism

$\mu(m) = \emptyset$ for all $m \in M$ {No students have been matched yet}
for $i = 1$ to $|W|$ **do**
  $M_w^i \leftarrow$ The set of all unmatched students with school w as their i^{th} ranked school.
  $W^i \leftarrow$ The set of all schools that are not at full capacity in round i.
  for $w \in W^i$ **do** Assign students $m \in M_w^i$ to school w in order of w's preference ordering until all students in M_w^i are matched or w has reached full capacity.
  end for
end for=0

We want to determine what properties this mechanism has. For the following questions, you must either explicitly prove the statement or disprove it with a counterexample and explanation. Your responses should make reference to the preferences $\succ$ of students and schools.

1. (5 pts) If students and schools report true preferences, does this mechanism output a Pareto optimal matching?
2. (5 pts) If students and schools report true preferences, does this mechanism output a stable matching?
3. (5 pts) For students, is this mechanism dominant strategy incentive compatible?

Distributional Domination (10 pts)

An action in a game can be strongly dominated by a *distribution* over other actions even if it is not dominated by any single other action. For example, consider the following game:

	a	b
A	$1, 6$	$0, 2$
B	$0, 4$	$1, 1$
C	$0.3, 5$	$0.3, 2$

The action C for the row player is not dominated by either actions A and B, but it is dominated by the mixed strategy that plays A with probability $1/2$ and B with probability $1/2$ (since this gets payoff in expectation at least $1/2$ against any strategy of the column player, which is greater than 0.3).

More generally, given a two player game with action sets A_1 and A_2 and utility functions u_1 and u_2, we say that an action $a_1 \in A_1$ for player 1 is distributionally strictly dominated if there exists a mixed strategy $p \in \Delta A_1$ such that for every action $a_2 \in A_2$, $u_1(p, a_2) > u_1(a_1, a_2)$.

Prove that each action $a_i \in A_1$ is a best response to some mixed strategy $q \in \Delta A_2$ if and only if a_i is not distributionally strictly dominated. In other words, if and only if a_i is not strictly distributionally dominated, there is some $q \in \Delta A_2$ such that:

$$a_i \in \arg\max_{a \in A_1} u_1(a, q).$$

HINT: Use the minimax theorem.