Algorithmic Game Theory: Problem Set 3
Due online via GradeScope before the start of class on Tuesday, February 28

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Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment. Ask questions on Piazza.

Problem 1) No Deterministic No Regret (10 points)

In class, we proved that the polynomial weights algorithm achieves “no regret” – that is, for arbitrary sequences of losses, it guarantees that the difference between the average loss achieved by the algorithm, and the average loss achieved by the best expert in hindsight is \( o(1) \) – tending to zero as \( T \to \infty \). Recall that the polynomial weights algorithm is randomized. Here we show that no deterministic algorithm can obtain the same guarantee.

1. (5 pts) Consider the algorithm “Follow the Leader” that always picks the expert that has the lowest cumulative loss so far – i.e. at day \( j \) it picks expert \( k \) such that:

\[
    k = \arg \min_{i} \sum_{t=1}^{j-1} \ell_{i,t}.
\]

(Suppose for concreteness that if there is a tie, the algorithm picks the min-loss expert with the smallest index). Show that Follow the Leader is not a no-regret algorithm. i.e. exhibit a sequence of losses in \([0,1]\) such that for all \( T \), the average regret of the algorithm is \( \Omega(1) \).

2. (5 pts) Prove that no deterministic experts algorithm can achieve \( o(1) \) regret against arbitrary losses in \([0,1]\) – i.e. that randomization is necessary to achieve a guarantee like that of the polynomial weights algorithm.

   Hint: Consider any fixed deterministic algorithm, and then place yourself in the role of an adversary who is trying to foil it. Can you, knowing which expert the algorithm is going to pick next, design a sequence of losses so that after \( T \) rounds, the best expert always has cumulative loss that is lower than the algorithm’s loss by at least \( T/2 \)?

Problem 2) Multiplayer Zero Sum Games (15 pts)

Consider a two person zero-sum game in which each player has \( k \) actions, defined by an \( k \times k \) matrix \( U \). We can think of the actions of both players as being the sets \( A_1 = A_2 = X = \{1, 2, \ldots, k\} \). Now consider the \( n \) person zero sum separable graphical game defined on a cycle of length \( n \) in which each person is simultaneously playing this zero-sum game with both their left neighbor and their right neighbor. With their right neighbor, they are taking the role of player 1 (the minimization player), and with their left
neighbor, they are taking the role of player 2 (the maximization player). Formally, if player \( i \) is playing strategy \( s_i \), then his utility is:

\[
u_i(s) = U(s_{i-1}, s_i) - U(s_i, s_{i+1})\]

where we understand the index \( i \mod n \). (i.e. \( 0 \equiv n \) and \( n + 1 \equiv 1 \)).

1. (5 pts) Show that for every \( \epsilon > 0 \) any such game must have a symmetric \( \epsilon \)-approximate Nash equilibrium – i.e. an \( \epsilon \)-approximate Nash equilibrium in which every player is playing the exact same mixed strategy. (Hint: Think about what would happen if all of these players repeatedly play the game using the polynomial weights algorithm).

2. (5 pts) Show that in any symmetric Nash equilibrium, the expected utility of each player is 0.

3. (5 pts) Suppose instead of a cycle, the players are on a path (i.e. players 1 and \( n \) only participate in a single zero sum game, with players 2 and \( n - 1 \) respectively). Is it still necessarily true that there is a Nash equilibrium in which every player has expected utility 0?

**Problem 3) Almost Zero Sum Games (15 pts)**

Consider a two player game \( G \) such that the two players have utility functions of the following form: for \( a_1 \in A_1, a_2 \in A_2 \):

\[
u_1(a_1, a_2) = f(a_1, a_2) + g(a_1) \quad \nu_2(a_1, a_2) = -f(a_1, a_2) + h(a_2)\]

where \( f, g, h \) can be arbitrary functions. Observe that \( G \) need not be zero-sum. Define an alternative game \( \hat{G} \) such that the two players have modified utility functions:

\[
\hat{u}_1(a_1, a_2) = f(a_1, a_2) + g(a_1) - h(a_2) \quad \hat{u}_2(a_1, a_2) = -f(a_1, a_2) + h(a_2) - g(a_1)
\]

Observe that \( \hat{G} \) is zero-sum.

1. (5 pts) Show that \( (p, q) \) is a Nash equilibrium for \( \hat{G} \) if and only if it is a Nash equilibrium for \( G \). In other words, \( G \) and \( \hat{G} \) are strategically equivalent.

2. (5 pts). Consider any sequence of action profiles \( (a^1, \ldots, a^T) \). Show that if this sequence of action profiles has regret \( \epsilon \) with respect to the player utility functions from \( G \), then it also has regret \( \epsilon \) with respect to the player utility functions from \( \hat{G} \).

3. (5 pts) Conclude that if two players play using the polynomial weights algorithm in \( G \), the empirical average of their play converges to Nash equilibrium (even though \( G \) is not 0 sum)

**Problem 4) Correlated Equilibria (15 pts)**

Recall the traffic light game:

<table>
<thead>
<tr>
<th></th>
<th>STOP</th>
<th>GO</th>
</tr>
</thead>
<tbody>
<tr>
<td>STOP</td>
<td>0,0</td>
<td>-100,100</td>
</tr>
<tr>
<td>GO</td>
<td>1,0</td>
<td></td>
</tr>
</tbody>
</table>

In class, we showed that this game has a correlated equilibrium (CE) of (GO, STOP)~ \( \frac{1}{2} \), (STOP, GO)~ \( \frac{1}{2} \) which attains better social welfare than any mixed Nash equilibrium (NE). Furthermore, the CE has the same social welfare as the game’s two pure NE while being more fair than either. We will now prove that the social welfare of CE can sometimes strictly exceed that of any NE, pure or mixed.
Consider modifying the traffic light game so that players prefer stopping when the opponent is also stopping (perhaps it allows for some pleasant conversation), and the collision cost is significantly ameliorated (due to improvements in air bag technology):

\[
\begin{array}{c|cc}
& \text{STOP} & \text{GO} \\
\hline
\text{STOP} & \left(\frac{2}{3}, \frac{2}{3}\right) & (0,1) \\
\text{GO} & (1,0) & (-1,-1)
\end{array}
\]

1. (5 pts) Find a welfare-maximizing NE of the new game (taking into account both mixed and pure strategies). You must prove that there is no NE with higher welfare.

2. (10 pts) Prove that the welfare-maximizing CE of this game has strictly higher social welfare than any NE.