

Algorithmic Game Theory: Problem Set 1

Due online via GradeScope before the start of class on Tuesday, February 1

Aaron Roth

Collaboration on problem sets is ok, but list everyone you worked with, and everyone must turn in their own assignment. Ask questions on Piazza.

Problem 1) Games with Infinite Action Sets (15 points)

John Nash proved that every game with finitely many players and finitely many actions has a Nash equilibrium in mixed strategies. These conditions are important!

- (a) Give an example of a 2 player game in which each player has infinitely many actions and your game has a Nash equilibrium. Precisely describe the equilibrium, and prove that it is a Nash equilibrium. (5 points)
- (b) Give an example of a 2 player game in which each player has infinitely many actions, and prove that your game does not have any Nash equilibrium. (10 points)

Hint: Don't forget about mixed strategies!

Problem 2) Iterated Elimination

Recall that in class we considered one way of solving a game: by iterated elimination of weakly dominated strategies. We can also consider iterated elimination of *strictly* dominated strategies. An action $a_i \in A_i$ is *strictly dominated* if $u_i(a_i, a_{-i}) < u_i(a'_i, a_{-i})$ for some $a'_i \in A$ and for all $a_{-i} \in A_{-i}$. (*i.e.* the inequality is always strict.)

We can write this method as an algorithm, which takes as input a set of n action sets A_1, \dots, A_n and a set of n utility functions u_1, \dots, u_n , where each u_i is a function $u_i : A_1 \times \dots \times A_n \rightarrow \mathbb{R}$.

Part 1 (10 points)

Consider the following 2 player game.

	<i>L</i>	<i>C</i>	<i>R</i>
<i>T</i>	2, 0	1, 1	4, 2
<i>M</i>	3, 4	1, 2	2, 3
<i>B</i>	1, 3	0, 2	3, 0

- (a) Which strategies survive iterated elimination of strictly dominated strategies? (2 points)
- (b) What are the pure strategy Nash equilibria of the game? (2 points)
- (c) Find a non-trivial (*i.e.* someone should be randomizing and not just playing a pure strategy) mixed-strategy Nash equilibrium of the game. (6 points)

Algorithm 1 Iterated Elimination of Strictly Dominated Strategies

IteratedElim($A_1, \dots, A_n, u_1, \dots, u_n$).

Initialize a counter $t = 0$

For each i , **Let** $B_i^t = A_i$

while TRUE **do**

For each i **let:**

$\text{Dom}_t^i = \{a_i \in B_i^t \text{ such that there exists } a'_i \in B_i^t \text{ such that for all } s \in B_1^t \times \dots \times B_n^t, u_i(a'_i, s_{-i}) > u_i(a_i, s_{-i})\}$

if There exists an i such that $\text{Dom}_t^i \neq \emptyset$ **then**

Let $B_i^{t+1} = B_i^t - \text{Dom}_t^i$

Update $t = t + 1$

else

Break

end if

end while

Return B_1^t, \dots, B_n^t .

Part 2 (35 points)

- (a) Prove that if only a single strategy profile s survives iterated elimination of weakly dominated strategies (*i.e.* if at the end for all i , $|B_i^t| = 1$ and $s_i \in B_i^t$ is the surviving action of player i) then s is a pure strategy Nash equilibrium of the game. (10 points)
- (b) Prove that if only a single strategy profile s survives iterated elimination of *strictly* dominated strategies, then it is the unique pure strategy Nash equilibrium of the game. (10 points)
- (c) Give an example of a game that has two pure strategy Nash equilibria, and depending on the order in which actions are chosen for elimination, *either* of them can be selected as the single surviving strategy profile when we apply iterated elimination of weakly dominated strategies. (5 points)
- (d) Consider the following game, “Guess Two-Thirds the Average”, in which each player submits a real number from 0 to 100, and the player whose submission is closest to two-thirds of the average submission wins. Formally, $|P| = n$, and for each player $i \in P$, $A_i = [0, 100]$. Given a collection of actions $a \in A$, let $w(a) = \frac{2}{3n} \sum_{i=1}^n a_i$, and let $\text{win}(a) = \arg \min_{i \in P} |a_i - w(a)|$ be the set of players whose submissions are closest to $2/3$ the average. The utility function for each player is such that $u_i(a) = 1/|\text{win}(a)|$ if $i \in \text{win}(a)$, and $u_i(a) = 0$ otherwise. Find the Nash equilibrium of this game via iterated elimination of weakly dominated strategies. Prove via induction that this is the only game state that survives iterated elimination. (10 points)