Algorithmic Game Theory: Problem Set 5
Due on Tuesday, April 13
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Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment.

Walrasian Equilibrium (20 pts)

1. We saw in lecture 14 that when buyers have unit demand valuations (or more generally, gross substitutes valuations) Walrasian equilibria always exist. In this problem you will give a simple example with two goods and two buyers showing that for general valuation functions, they need not exist. Specifically, consider a market consisting of two goods: \( G = \{ PB, J \} \). Describe two valuation functions \( v_1 : 2^G \to \mathbb{R}_{\geq 0} \), \( v_2 : 2^G \to \mathbb{R}_{\geq 0} \) such that Walrasian equilibria do not exist in the market with two buyers with valuations \( v_1 \) and \( v_2 \). Prove that no Walrasian equilibrium exists in your example. (10 pts).

2. Walrasian equilibria need not be unique. Consider a market in which there are multiple Walrasian equilibria, with two such equilibria denoted as price/allocation pairs \((p, (S_1, \ldots, S_n))\) and \((p', (S'_1, \ldots, S'_n))\) respectively. Prove that in this case, we can swap the price vectors while preserving the equilibrium property — i.e. prove that \((p, (S'_1, \ldots, S'_n))\) and \((p', (S_1, \ldots, S_n))\) are also Walrasian equilibria. (10 pts)

Generalized Groves Mechanism (10 pts)

In class, we saw that the Groves mechanism with allocation rule: \( x(v) = \arg \max_{a \in A} \sum_{i=1}^{n} v_i(a) \) and payment rule \( p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(a^*) \) where \( a^* = x(v) \) and \( h_i \) is an arbitrary function independent of \( v_i \) is truthful. Suppose instead we have a generalized allocation rule:

\[
x(v) = \arg \max_{a \in A} \left( \sum_{i=1}^{n} \alpha_i v_i(a) + \beta_a \right)
\]

where \( \alpha_i > 0 \) are positive weights corresponding to individuals and \( \beta_a \) are weights corresponding to alternatives \( a \in A \). Show that this allocation rule can be paired with a payment rule such that the resulting mechanism is truthful.

Public Projects Auction (15 pts)

Recall the public projects auction, in which a city is deciding whether or not to build a bridge at cost \( C > 0 \). In this case, \( A = \{ \text{yes}, \text{no} \} \). This is a single parameter domain, in which \( w(\text{yes}) = 1 \) and \( w(\text{no}) = 0 \) and an individual with valuation \( v_i \in \mathbb{R}^+ \) has value \( v_i(a) = v_i \cdot w(a) \) for outcome \( a \). Consider the (generalized)
VCG mechanism which has allocation rule $x(v) = \arg \max_{a \in A} \sum_{i=1}^{n} v_i(a) - w(a) \cdot C$ (i.e. $\beta_{yes} = -C$) and payment rule

$$p_i(v) = \left( \sum_{j \neq i} v_j(a^*_{-i}) - w(a^*_{-i})C \right) - \left( \sum_{j \neq i} v_j(a^*) - w(a^*)C \right)$$

where $a^* = x(v)$ and $a^*_{-i} = x(v_{-i})$. This VCG mechanism builds the bridge if and only if $\sum_{i=1}^{n} v_i \geq C$.

1. Show that if the bridge is built, the total payments are enough to cover the cost of the bridge (i.e. $\sum_{i=1}^{n} p_i(v) \geq C$) if and only if $\sum_{i=1}^{n} v_i = C$. (10 pts)

2. Show that although the VCG mechanism is dominant strategy truthful, it is not robust to deviations by coalitions. Specifically, describe a set of valuation functions such that there is a pair of individuals who can coordinate on a joint deviation from truth-telling that results in both of them obtaining higher utility than they would if they reported their valuations truthfully. (5 pts)