Algorithmic Game Theory: Problem Set 6
Due on Tuesday, April 28
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Remember you can work together on problem sets, but list everyone you worked with, and everyone turn in their own assignment.

Efficient Implementation of VCG (15 pts)

Consider an auction setting in which the auctioneer is selling $m$ identical copies of some good. Buyers can be assigned multiple copies of the good, and have valuation functions $v_i$ which map the number of copies of the good they receive to their total value: $v_i : \{0, 1, \ldots, m\} \to \mathbb{R}$. Write $v_{i,j}$ for buyer $i$'s marginal value for the $j$'th copy of a good – i.e. let $v_i(j) = \sum_{j=1}^{k} v_{i,j}$. Suppose further that each buyer $i$ has decreasing marginal utility for these goods – in other words, $v_{i,1} \geq v_{i,2} \geq \ldots \geq v_{i,m}$. For simplicity you can assume that all $v_{i,j}$ are distinct.

1. Give a simple greedy algorithm for implementing the allocation rule of the VCG mechanism in this setting. Prove that your algorithm optimizes welfare. Does this algorithm still work if buyer valuations do not satisfy the decreasing marginal utility property? (10 pts)

2. Give a simple description of the payment of a bidder in the VCG mechanism as a sum of marginal valuations reported by the other bidders. (5 pts)

Knapsack Auctions (15 pts)

1. In class, we showed a 2-approximation algorithm for the knapsack auction, and proved that it was a monotone allocation rule. Give an explicit description of the payment rule which makes it a dominant strategy to report valuations truthfully. (5 pts)

2. Now consider a variant of the knapsack auction in which each bidder $i$ has to report both his value $v_i$ and his size $w_i$. This is no longer a single parameter domain. An allocation rule $x(v', w')$ now specifies the amount of capacity allocated to each bidder as a function of their reported bids and sizes. The utility of buyer $i$ is defined to be $v_i - p_i(v', w')$ if she gets her full capacity – i.e. if $x_i(v', w') \geq w_i$, and to be $-p_i(v, w)$ otherwise (i.e. she gets no value for getting capacity less than her full size). Consider the 2-approximation we considered in class that simply takes the reported sizes $w'$ at face value (and define $x_i(v', w') = w'$ if the algorithm allocates buyer $i$ with these reports, and $x_i(v', w') = 0$ otherwise), and uses the same payment rule computed above. Does this mechanism make reporting true values and sizes a dominant strategy? Prove it if so, or give an explicit counter-example. (10 pts)
Approximating Welfare with Posted Pricings (15 pts)

Consider a buyer with a single item for sale, that must be priced with some price \( p \in [0, 1] \). \( n \) buyers arrive one at a time, and the first buyer \( i \) (if any) with valuation \( v_i \geq p \) buys the item. Recall that we showed that when each buyer has their valuation drawn independently \( v_i \sim D_i \) from a known distribution, then we can obtain half of the optimal welfare in expectation by setting \( p = \frac{1}{2} \mathbb{E}[V^*] \), where \( V^* = \max_i v_i \). In this problem, we will show that an alternative procedure fails to get any approximation to the optimal welfare.

Here is the alternative:

**MostValuableBidder:**

1. Select a buyer \( i^* \) with highest expected value \( i^* = \arg \max_{v_i \sim D_i} \mathbb{E}[v_i] \), breaking ties arbitrarily.
2. Set a personalized price \( p_{i^*} = 0 \) for buyer \( i^* \) and set \( p_j = \infty \) for all \( j \neq i^* \). (i.e. just give the item to buyer \( i^* \) for free).

We now consider the following setting in which for all \( i \), \( D_i = D^\alpha \) is defined as follows:

\[
\Pr_{v \sim D^\alpha}[v = 0] = 1 - \alpha \quad \Pr_{v \sim D^\alpha}[v = 1] = \alpha
\]

1. What is the expected welfare obtained by **MostValuableBidder** as a function of \( \alpha \) and \( n \)? (5 pts)
2. Derive an exact expression for \( \mathbb{E}[V^*] \) as a function of \( \alpha \) and \( n \). Conclude that **MostValuableBidder** has no finite approximation guarantee to the optimal welfare. (10 pts).