Algorithmic Game Theory: Problem Set 1
Due online via GradeScope before the start of class on Tuesday, February 4
Aaron Roth

Collaboration on problem sets is ok, but list everyone you worked with, and everyone must turn in their own assignment. Ask questions on Piazza.

Problem 1) Games with Infinite Action Sets (15 points)

John Nash proved that every game with finitely many players and finitely many actions has a Nash equilibrium in mixed strategies. These conditions are important!

(a) Give an example of a 2 player game in which each player has infinitely many actions and your game has a Nash equilibrium. Precisely describe the equilibrium, and prove that it is a Nash equilibrium. (5 points)

(b) Give an example of a 2 player game in which each player has infinitely many actions, and prove that your game does not have any Nash equilibrium. (10 points)
Hint: Don’t forget about mixed strategies!

Problem 2) Iterated Elimination

Recall that in class we considered one way of solving a game: by iterated elimination of weakly dominated strategies. We can also consider iterated elimination of strictly dominated strategies. An action $a_i \in A_i$ is strictly dominated if $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$ for some $a_i' \in A$ and for all $a_{-i} \in A_{-i}$. (i.e. the inequality is always strict.)

We can write this method as an algorithm, which takes as input a set of $n$ action sets $A_1, \ldots, A_n$ and a set of $n$ utility functions $u_1, \ldots, u_n$, where each $u_i$ is a function $u_i : A_1 \times \ldots \times A_n \to \mathbb{R}$.

Part 1 (10 points)

Consider the following 2 player game.

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$C$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>2,0</td>
<td>1,1</td>
<td>4,2</td>
</tr>
<tr>
<td>$M$</td>
<td>3,4</td>
<td>1,2</td>
<td>2,3</td>
</tr>
<tr>
<td>$B$</td>
<td>1,3</td>
<td>0,2</td>
<td>3,0</td>
</tr>
</tbody>
</table>

(a) Which strategies survive iterated elimination of strictly dominated strategies? (2 points)

(b) What are the pure strategy Nash equilibria of the game? (2 points)

(c) Find a non-trivial (i.e. someone should be randomizing and not just playing a pure strategy) mixed-strategy Nash equilibrium of the game. (6 points)
Algorithm 1 Iterated Elimination of Strictly Dominated Strategies

IteratedElim(A₁, ..., Aₙ, u₁, ..., uₙ).

Initialize a counter t = 0
For each i, Let Bₜᵢ = Aᵢ
while TRUE do
    For each i let:
        Domₜᵢ = \{aᵢ ∈ Bₜᵢ such that there exists aᵢ' ∈ Bₜᵢ such that for all s ∈ Bₜ₁ × ... × Bₜₙ, uᵢ(aᵢ', sᵢ) > uᵢ(aᵢ, sᵢ)\}
    if There exists an i such that Domₜᵢ ≠ ∅ then
        Let Bₜᵢ₊₁ = Bₜᵢ - Domₜᵢ
        Update t = t + 1
    else
        Break
    end if
end while
Return Bₜ₁, ..., Bₜₙ.

Part 2 (35 points)

(a) Prove that if only a single strategy profile s survives iterated elimination of weakly dominated strategies (i.e. if at the end for all i, |Bₜᵢ| = 1 and sᵢ ∈ Bₜᵢ is the surviving action of player i) then s is a pure strategy Nash equilibrium of the game. (10 points)

(b) Prove that if only a single strategy profile s survives iterated elimination of strictly dominated strategies, then it is the unique pure strategy Nash equilibrium of the game. (10 points)

(c) Give an example of a game that has two pure strategy Nash equilibria, and depending on the order in which actions are chosen for elimination, either of them can be selected as the single surviving strategy profile when we apply iterated elimination of weakly dominated strategies. (5 points)

(d) Consider the following game, “Guess Two-Thirds the Average”, in which each player submits a real number from 0 to 100, and the player whose submission is closest to two-thirds of the average submission wins. Formally, |P| = n, and for each player i ∈ P, Aᵢ = [0, 100]. Given a collection of actions a ∈ A, let w(a) = 2/n \sumᵢ₌₁ⁿ aᵢ, and let win(a) = arg minᵢ∈P |aᵢ - w(a)| be the set of players whose submissions are closest to 2/3 the average. The utility function for each player is such that uᵢ(a) = 1/|win(a)| if i ∈ win(a), and uᵢ(a) = 0 otherwise. Find the Nash equilibrium of this game via iterated elimination of weakly dominated strategies. Prove via induction that this is the only game state that survives iterated elimination. (10 points)