Chapter 16

Isometries, Local Isometries, Riemannian Coverings and Submersions, Killing Vector Fields

16.1 Isometries and Local Isometries

Recall that a *local isometry* between two Riemannian manifolds M and N is a smooth map $\varphi\colon M \to N$ so that

$$\langle (d\varphi)_p(u), (d\varphi_p)(v)\rangle_{\varphi(p)} = \langle u, v\rangle_p,$$

for all $p \in M$ and all $u, v \in T_pM$. An *isometry* is a local isometry and a diffeomorphism.

By the inverse function theorem, if $\varphi\colon M \to N$ is a local isometry, then for every $p \in M$, there is some open subset $U \subseteq M$ with $p \in U$ so that $\varphi \restriction U$ is an isometry between U and $\varphi(U)$.

Also recall that if $\varphi\colon M \to N$ is a diffeomorphism, then for any vector field X on M, the vector field $\varphi_* X$ on N (called the *push-forward* of X) is given by

$$(\varphi_* X)_q = d\varphi_{\varphi^{-1}(q)} X(\varphi^{-1}(q)), \qquad \text{for all } q \in N,$$

or equivalently, by

$$(\varphi_* X)_{\varphi(p)} = d\varphi_p X(p), \qquad \text{for all } p \in M.$$

For any smooth function $h\colon N \to \mathbb{R}$, for any $q \in N$, we have

$$X_*(h)_q = X(h \circ \varphi)_{\varphi^{-1}(q)},$$

or

$$X_*(h)_{\varphi(p)} = X(h \circ \varphi)_p.$$

It is natural to expect that isometries preserve all "natural" Riemannian concepts and this is indeed the case. We begin with the Levi-Civita connection.

Proposition 16.1. *If* $\varphi\colon M \to N$ *is an isometry, then*

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_* X}(\varphi_* Y), \qquad \textit{for all } X, Y \in \mathfrak{X}(M),$$

where $\nabla_X Y$ *is the Levi-Civita connection induced by the metric on* M *and similarly on* N.

As a corollary of Proposition 16.1, the curvature induced by the connection is preserved; that is

$$\varphi_* R(X, Y)Z = R(\varphi_* X, \varphi_* Y)\varphi_* Z,$$

as well as the parallel transport, the covariant derivative of a vector field along a curve, the exponential map, sectional curvature, Ricci curvature and geodesics.

Actually, all concepts that are local in nature are preserved by local diffeomorphisms! So, except for the Levi-Civita connection and the Riemann tensor on vectors, all the above concepts are preserved under local diffeomorphisms.

Proposition 16.2. *If $\varphi\colon M \to N$ is a local isometry, then the following concepts are preserved:*

(1) The covariant derivative of vector fields along a curve γ; that is

$$d\varphi_{\gamma(t)}\frac{DX}{dt} = \frac{D\varphi_* X}{dt},$$

for any vector field X along γ, with $(\varphi_* X)(t) = d\varphi_{\gamma(t)}Y(t)$, for all t.

(2) Parallel translation along a curve. If P_γ denotes parallel transport along the curve γ and if $P_{\varphi\circ\gamma}$ denotes parallel transport along the curve $\varphi \circ \gamma$, then

$$d\varphi_{\gamma(1)} \circ P_\gamma = P_{\varphi\circ\gamma} \circ d\varphi_{\gamma(0)}.$$

(3) *Geodesics. If γ is a geodesic in M, then $\varphi \circ \gamma$ is a geodesic in N. Thus, if γ_v is the unique geodesic with $\gamma(0) = p$ and $\gamma_v'(0) = v$, then*

$$\varphi \circ \gamma_v = \gamma_{d\varphi_p v},$$

wherever both sides are defined. Note that the domain of $\gamma_{d\varphi_p v}$ may be strictly larger than the domain of γ_v. For example, consider the inclusion of an open disc into $\mathbb{R}^2$.

(4) *Exponential maps. We have*

$$\varphi \circ \exp_p = \exp_{\varphi(p)} \circ d\varphi_p,$$

wherever both sides are defined.

(5) *Riemannian curvature tensor. We have*

$$d\varphi_p R(x, y)z = R(d\varphi_p x, d\varphi_p y)d\varphi_p z, \quad \text{for all } x, y, z \in T_pM.$$

(6) Sectional, Ricci, and Scalar curvature. We have

$$K(d\varphi_p x, d\varphi_p y) = K(x, y)_p,$$

for all linearly independent vectors $x, y \in T_pM$*;*

$$\mathrm{Ric}(d\varphi_p x, d\varphi_p y) = \mathrm{Ric}(x, y)_p$$

for all $x, y \in T_pM$*;*

$$S_M = S_N \circ \varphi.$$

where S_M *is the scalar curvature on* M *and* S_N *is the scalar curvature on* N*.*

A useful property of local diffeomorphisms is stated below. For a proof, see O'Neill [44] (Chapter 3, Proposition 62):

Proposition 16.3. *Let* $\varphi, \psi \colon M \to N$ *be two local isometries. If* M *is connected and if* $\varphi(p) = \psi(p)$ *and* $d\varphi_p = d\psi_p$ *for some* $p \in M$*, then* $\varphi = \psi$*.*

16.2 Riemannian Covering Maps

The notion of covering map discussed in Section 7.3 (see Definition 7.8) can be extended to Riemannian manifolds.

Definition 16.1. If M and N are two Riemannian manifold, then a map $\pi\colon M \to N$ is a *Riemannian covering* iff the following conditions hold:

(1) The map π is a smooth covering map.

(2) The map π is a local isometry.

Recall from Section 7.3 that a covering map is a local diffeomorphism.

A way to obtain a metric on a manifold M is to pull-back the metric g on a manifold N along a local diffeomorphism $\varphi\colon M \to N$ (see Section 11.2).

If φ is a covering map, then it becomes a Riemannian covering map.

Proposition 16.4. *Let $\pi\colon M \to N$ be a smooth covering map. For any Riemannian metric g on N, there is a unique metric $\pi^* g$ on M, so that π is a Riemannian covering.*

In general, if $\pi\colon M \to N$ is a smooth covering map, a metric on M does not induce a metric on N such that π is a Riemannian covering.

However, if N is obtained from M as a quotient by some suitable group action (by a group G) on M, then the projection $\pi\colon M \to M/G$ is a Riemannian covering.

Because a Riemannian covering map is a local isometry, we have the following useful result.

Proposition 16.5. *Let $\pi\colon M \to N$ be a Riemannian covering. Then, the geodesics of (M, g) are the projections of the geodesics of (N, h) (curves of the form $\pi \circ \gamma$, where γ is a geodesic in N), and the geodesics of (N, h) are the liftings of the geodesics of (M, h) (curves γ in N such that $\pi \circ \gamma$ is a geodesic of (M, h)).*

As a corollary of Proposition 16.4 and Theorem 7.12, every connected Riemannian manifold M has a simply connected covering map $\pi\colon \widetilde{M} \to M$, where π is a Riemannian covering.

Furthermore, if $\pi\colon M \to N$ is a Riemannian covering and $\varphi\colon P \to N$ is a local isometry, it is easy to see that its lift $\widetilde{\varphi}\colon P \to M$ is also a local isometry.

In particular, the deck-transformations of a Riemannian covering are isometries.

In general, a local isometry is not a Riemannian covering. However, this is the case when the source space is complete.

Proposition 16.6. *Let* $\pi\colon M \to N$ *be a local isometry with* N *connected. If* M *is a complete manifold, then* π *is a Riemannian covering map.*

16.3 Riemannian Submersions

Let $\pi\colon M \to B$ be a surjective submersion between two Riemannian manifolds (M, g) and (B, h).

For every $b \in B$, the fibre $\pi^{-1}(b)$ is a Riemannian submanifold of M, and for every $p \in \pi^{-1}(b)$, the tangent space $T_p\pi^{-1}(b)$ to $\pi^{-1}(b)$ at p is $\mathrm{Ker}\, d\pi_p$.

The tangent space T_pM to M at p splits into the two components

$$T_pM = \mathrm{Ker}\, d\pi_p \oplus (\mathrm{Ker}\, d\pi_p)^{\perp},$$

where $\mathcal{V}_p = \mathrm{Ker}\, d\pi_p$ is the *vertical subspace* of T_pM and $\mathcal{H}_p = (\mathrm{Ker}\, d\pi_p)^{\perp}$ (the orthogonal complement of $\mathcal{V}_p$ with respect to the metric g_p on T_pM) is the *horizontal subspace* of T_pM.

Any tangent vector $u \in T_pM$ can be written uniquely as

$$u = u_{\mathcal{H}} + u_{\mathcal{V}},$$

with $u_{\mathcal{H}} \in \mathcal{H}_p$ and $u_{\mathcal{V}} \in \mathcal{V}_p$.

Because π is a submersion, $d\pi_p$ gives a linear isomorphism between $\mathcal{H}_p$ and T_bB.

If $d\pi_p$ is an isometry, then most of the differential geometry of B can be studied by "lifting" from B to M.

Definition 16.2. A map $\pi\colon M \to B$ between two Riemannian manifolds (M, g) and (B, h) is a *Riemannian submersion* if the following properties hold:

(1) The map π is surjective and a smooth submersion.

(2) For every $b \in B$ and every $p \in \pi^{-1}(b)$, the map $d\pi_p$ is an isometry between the horizontal subspace $\mathcal{H}_p$ of T_pM and T_bB.

We will see later that Riemannian submersions arise when B is a reductive homogeneous space, or when B is obtained from a free and proper action of a Lie group acting by isometries on B.

Every vector field X on B has a unique *horizontal lift* $\overline{X}$ on M, defined such that for every $p \in \pi^{-1}(b)$,

$$\overline{X}(p) = (d\pi_p)^{-1} X(b).$$

Since $d\pi_p$ is an isomorphism between $\mathcal{H}_p$ and T_pB, the above condition can be written

$$d\pi \circ \overline{X} = X \circ \pi,$$

which means that $\overline{X}$ and X are π-related (see Definition 6.5).

The following proposition is proved in O'Neill [44] (Chapter 7, Lemma 45) and Gallot, Hulin, Lafontaine [23] (Chapter 2, Proposition 2.109).

Proposition 16.7. *Let $\pi\colon M \to B$ be a Riemannian submersion between two Riemannian manifolds (M, g) and (B, h).*

(1) For any two vector fields $X, Y \in \mathfrak{X}(B)$, we have

(a) $\langle \overline{X}, \overline{Y} \rangle = \langle X, Y \rangle \circ \pi$.

(b) $[\overline{X}, \overline{Y}]_{\mathcal{H}} = \overline{[X, Y]}$.

(c) $(\nabla_{\overline{X}} \overline{Y})_{\mathcal{H}} = \overline{\nabla_X Y}$, *where ∇ is the Levi–Civita connection on M.*

(2) If γ is a geodesic in M such that $\gamma'(0)$ is a horizontal vector, then γ is horizontal geodesic in M (which means that $\gamma'(t)$ is a horizontal vector for all t), and $c = \pi \circ \gamma$ is a geodesic in B of the same length than γ.

(3) For every $p \in M$, if c is a geodesic in B such that $c(0) = \pi(p)$, then for some ϵ small enough, there is a unique horizonal lift γ of the restriction of c to $[-\epsilon, \epsilon]$, and γ is a geodesic of M.

(4) If M is complete, then B is also complete.

An example of a Riemannian submersion is $\pi\colon S^{2n+1} \to \mathbb{CP}^n$, where S^{2n+1} has the canonical metric and $\mathbb{CP}^n$ has the Fubini–Study metric.

Remark: It shown in Petersen [45] (Chapter 3, Section 5), that the connection $\nabla_{\overline{X}}\overline{Y}$ on M is given by

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]_{\mathcal{V}}.$$

16.4 Isometries and Killing Vector Fields ⊛

Recall from Section **??** that if X is a vector field on a manifold M, then for any $(0, q)$-tensor $S \in \Gamma(M, (T^*)^{\otimes q}(M))$, the Lie derivative $L_X S$ of S with respect to X is defined by

$$(L_X S)_p = \frac{d}{dt}(\Phi_t^* S)_p \bigg|_{t=0}, \quad \in M,$$

where Φ_t is the local one-parameter group associated with X, and that by Proposition **??**,

$$\begin{aligned}(L_X S)(X_1, \ldots, X_q) &= X(S(X_1, \ldots, X_q)) \\ &\quad - \sum_{i=1}^{q} S(X_1, \ldots, [X, X_i], \ldots, X_q),\end{aligned}$$

for all $X_1, \ldots, X_q \in \mathfrak{X}(M)$.

In particular, if $S = g$ (the metric tensor), we get

$$L_X g(Y, Z) = X(\langle Y, Z\rangle) - \langle [X, Y], Z\rangle - \langle Y, [X, Z]\rangle,$$

where we write $\langle X, Y\rangle$ and $g(X, Y)$ interchangeably.

If Φ_t is an isometry (on its domain), then $\Phi_t^*(g) = g$, so $L_X g = 0$.

In fact, we have the following result proved in O'Neill [44] (Chapter 9, Proposition 23).

Proposition 16.8. *For any vector field X on a Riemannian manifold (M, g), the diffeomorphisms Φ_t induced by the flow Φ of X are isometries (on their domain) iff $L_X g = 0$.*

Informally, Proposition 16.8 says that $L_X g$ measures how much the vector field X changes the metric g.

Definition 16.3. Given a Riemannian manifold (M, g), a vector field X is a *Killing vector field* iff the Lie derivative of the metric vanishes; that is, $L_X g = 0$.

Recall from Section **??** (see Proposition **??**) that the covariant derivative $\nabla_X g$ of the Riemannian metric g on a manifold M is given by

$$\nabla_X(g)(Y, Z) = X(\langle Y, Z\rangle) - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle,$$

for all $X, Y, Z \in \mathfrak{X}(M)$, and that the connection ∇ on M is compatible with g iff $\nabla_X(g) = 0$ for all X.

Also, the covariant derivative ∇X of a vector field X is the $(1,1)$-tensor defined so that

$$(\nabla X)(Y) = \nabla_Y X.$$

The above facts imply the following Proposition.

Proposition 16.9. *Let (M, g) be a Riemannian manifold and let ∇ be the Levi–Civita connection on M induced by g. For every vector field X on M, the following conditions are equivalent:*

(1) X is a Killing vector field; that is, $L_X g = 0$.

(2) $X(\langle Y, Z\rangle) = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle$ for all $Y, Z \in \mathfrak{X}(M)$.

(3) $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all $Y, Z \in \mathfrak{X}(M)$; that is, ∇X is skew-adjoint relative to g.

Condition (3) shows that any parallel vector field is a Killing vector field.

Remark: It can be shown that if γ is any geodesic in M, then the restriction X_γ of X to γ is a Jacobi field (see Section 14.5), and that $\langle X, \gamma' \rangle$ is constant along γ (see O'Neill [44], Chapter 9, Lemma 26).

Since the Lie derivative L_X is $\mathbb{R}$-linear in X and since

$$[L_X, L_Y] = L_{[X,Y]},$$

the Killing vector fields on M form a Lie subalgebra $\mathcal{K}i(M)$ of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M.

However, unlike $\mathfrak{X}(M)$, the Lie algebra $\mathcal{K}i(M)$ is finite-dimensional.

In fact, the Lie subalgebra $c\mathcal{K}i(M)$ of complete Killing vector fields is anti-isomorphic to the Lie algebra $\mathfrak{i}(M)$ of the Lie group $\mathrm{Isom}(M)$ of isometries of M (see Section 11.2 for the definition of $\mathrm{Isom}(M)$).

The following result is proved in O'Neill [44] (Chapter 9, Lemma 28) and Sakai [49] (Chapter III, Lemma 6.4 and Proposition 6.5).

Proposition 16.10. *Let (M, g) be a connected Riemannian manifold of dimension n (equip-ped with the Levi–Civita connection on M induced by g). The Lie algebra $\mathcal{K}i(M)$ of Killing vector fields on M has dimension at most $n(n+1)/2$.*

We also have the following result proved in O'Neill [44] (Chapter 9, Proposition 30) and Sakai [49] (Chapter III, Corollary 6.3).

Proposition 16.11. *Let (M, g) be a Riemannian manifold of dimension n (equipped with the Levi–Civita connection on M induced by g). If M is complete, then every Killing vector fields on M is complete.*

The relationship between the Lie algebra $\mathfrak{i}(M)$ and Killing vector fields is obtained as follows.

For every element X in the Lie algebra $\mathfrak{i}(M)$ of $\mathrm{Isom}(M)$ (viewed as a left-invariant vector field), define the vector field X^+ on M by

$$X^+(p) = \frac{d}{dt}(\varphi_t(p))\bigg|_{t=0}, \quad p \in M,$$

where $t \mapsto \varphi_t = \exp(tX)$ is the one-parameter group associated with X.

Because φ_t is an isometry of M, the vector field X^+ is a Killing vector field, and it is also easy to show that (φ_t) is the one-parameter group of X^+.

Since φ_t is defined for all t, the vector field X^+ is complete. The following result is shown in O'Neill [44] (Chapter 9, Proposition 33).

Theorem 16.12. *Let (M, g) be a Riemannian manifold (equipped with the Levi–Civita connection on M induced by g). The following properties hold:*

(1) The set $c\mathcal{K}i(M)$ of complete Killing vector fields on M is a Lie subalgebra of the Lie algebra $\mathcal{K}i(M)$ of Killing vector fields.

(2) The map $X \mapsto X^+$ is a Lie anti-isomorphism between $\mathfrak{i}(M)$ and $c\mathcal{K}i(M)$, which means that

$$[X^+, Y^+] = -[X, Y]^+, \quad X, Y \in \mathfrak{i}(M).$$

For more on Killing vector fields, see Sakai [49] (Chapter III, Section 6).

In particular, complete Riemannian manifolds for which $\mathfrak{i}(M)$ has the maximum dimension $n(n+1)/2$ are characterized.