Preview of Lecture 03.14

On 03.14, we will discuss another interesting aspect of the expressive power of polyadic quantification theory. We write $\mathbb{Z}^+$ for the set of positive integers \{1, 2, 3, \ldots\}. The spectrum of a schema $S$ (written $\text{Spec}(S)$) is defined as follows.

$$\text{Spec}(S) = \{n \in \mathbb{Z}^+ \mid \text{mod}(S, n) \neq \emptyset\}.$$ 

Recall the schema $SG \land 1\text{reg}$ which defines the collection of 1-regular simple graphs. We have already noticed that $\text{Spec}(SG \land 1\text{reg})$ is the set of even numbers, that is, $\text{Spec}(SG \land 1\text{reg}) = \{2^i \mid i \in \mathbb{Z}^+\}$.

Let’s look at another important class of graphs, namely, equivalence relations, and see how they can be put to use in generating schemata with a wide range of spectra. A graph $A$ is an equivalence relation if and only if $A \models \text{Eq}$, where $\text{Eq}$ is the conjunction of the following schemata.

- **Refl:** $(\forall x)Lxx$
- **Sym:** $(\forall x)(\forall y)(Lxy \supset Lyx)$
- **Trans:** $(\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz))$

Now suppose we’d like to construct a schema $S$ such that

- $S$ implies $\text{Eq}$, and
- $\text{Spec}(S) = \{3i + 1 \mid i \in \mathbb{Z}^+ \cup \{0\}\}$.

The easiest way to meet the first condition is to formulate $S$ as a conjunction, one conjunct of which is $\text{Eq}$ itself. But what more should we say? Well, the universe $U^A$ of an equivalence relation $A$ is partitioned into mutually disjoint equivalence classes by the relation $L^A$; for each $a \in U^A$, the equivalence class $\hat{a}$ of $a$, is $\{b \in U^A \mid \langle a, b \rangle \in L^A\}$. Now if we can construct a schema $T$ that says every equivalence class but one is of size three, and that the exceptional equivalence class is of size one, then we may take $S$ to be the conjunction of $\text{Eq}$ and $T$. The following schema $T$ does the job.

$$(\exists x)(\forall t)((\forall y)(Lty \supset y = t) \equiv x = t) \land$$

$$(\forall z)((\exists r)(r \neq z \land Lrz) \supset$$

$$(\exists v)(\exists w)(v \neq z \land v \neq w \land w \neq z \land (\forall u)(Luz \equiv (u = z \lor u = v \lor u = w))))$$

We will go on to explore further examples of spectra. Note that in general it is not the case that $\text{Spec}(\neg S) = \mathbb{Z}^+ - \text{Spec}(S)$. Convince yourself by constructing some examples where the equation fails! Can you think of examples where the equation holds? Can you think of a general condition on $\text{Spec}(S)$ that guarantees the failure of the equation? We will discuss these questions in class on Monday.