# Lecture 11.09.08

Logic is the science of truth. Truth arises from relations between language and the world. Logic provides mathematical models of such relations. In logic, we study formal languages $L$ and structures $A$ which play the role of language and the world. The central relation is truth of a sentence $\varphi \in L$ in a structure $A$ (we write this as $A \models \varphi$, and we say $\varphi$ is true in $A$, or $A$ satisfies $\varphi$). In terms of this relation, we define the notion of logical consequence: if $\Sigma \subseteq L$ is a set of sentences and $\varphi \in L$ is a sentence we say $\varphi$ is a logical consequence of $\Sigma$ (and write $\Sigma \models \varphi$), if and only if,

$$
\text{for all structures } A, \text{ if } A \models \Sigma, \text{ then } A \models \varphi.
$$

We can also define the set $V$ of valid sentences of $L$, namely, $\varphi \in L$ is valid, if and only if, for every structure $A$, $A \models \varphi$.

One important example of a formal language is first order logic. This language suffices for the formalization of large tracts of scientific discourse. A part of this course will be devoted to answering two interesting epistemological (even technological) questions concerning first order validity.

1. Can we find out that a sentence of first order logic is valid, if in fact it is?
2. Can we determine whether or not a sentence of first order logic is valid?

The answers to both these questions emerged from work of Kurt Gödel, Alonzo Church, and Alan Turing.

1. Gödel’s Completeness Theorem: $V$ is semi-decidable.
2. Church-Turing Theorem: $V$ is not decidable.

Reading: Enderton, Sections 1.1 & 1.2
Exercises: Enderton, Exercise 1.2.10 (Please devote some attention to this exercise, since I would like to go over it during our next class meeting.)

For those who would like to proceed ahead, our next reading will be sections 1.5 and 1.7; exercises 1.7.1-1.7.3.
2 Lecture 11.09.13

Some definitions:
1. \( O = \{ A_i \mid i \in N \} = \) the set of sentence letters;
2. \( S = \) the set of sentences generated from \( O \) using the sentential connectives;
3. \( H = \{ h : O \rightarrow \{ T, F \} \} = \) the set of truth assignments;
4. for \( \alpha \in S \) and \( h \in H \), \( h \models \alpha \), if and only if, \( \exists (h(\alpha) = T \ (h \text{ satisfies } \alpha)) \);
5. for \( \Sigma \subseteq S \) and \( h \in H \), \( h \models \Sigma \), if and only if, for all \( \varphi \in \Sigma \), \( h \models \varphi \) (if \( h \) satisfies \( \Sigma \));
6. for \( \Sigma \subseteq S \) and \( \alpha \in S \), \( \Sigma \models \alpha \) if and only if for all \( h \in H \), if \( h \models \Sigma \), then \( h \models \alpha \). (\( \alpha \) is a logical consequence of \( \Sigma \));
7. For \( \Sigma \subseteq S \) let \( Cn(\Sigma) = \{ \varphi \mid \Sigma \models \varphi \} \). (The set of logical consequences of \( \Sigma \), aka the theory axiomatized by \( \Sigma \));
8. \( \Sigma \) is equivalent to \( \Gamma \), if and only if, \( Cn(\Gamma) = Cn(\Sigma) \).
9. \( \Sigma \) is independent, if and only if, for all \( \alpha \in \Sigma \), \( \{ \Sigma - \{ \alpha \} \} \not\models \alpha \).

We discussed the inductive definition of \( S \) (consult Enderton, Section 1.4 for additional discussion of inductive definitions).

1.2.10(a) Show that if \( \Sigma \subseteq S \) is finite, then there is a \( \Delta \subseteq \Sigma \) such that \( \Sigma \) is equivalent to \( \Delta \) and \( \Delta \) is independent. We proceeded by induction on the size of \( \Sigma \) and noted the importance of considering the case where \( \Sigma \) is empty, since any set of tautologies is equivalent to the \( \emptyset \). We reduced the induction step to showing that for every \( \Sigma \subseteq S \) and \( \alpha \in S \), if \( (\Sigma - \{ \alpha \}) \models \alpha \), then \( (\Sigma - \{ \alpha \}) \) is equivalent to \( \Sigma \).

It follows at once from the lemma below that for every \( \Sigma \subseteq S \) and \( \alpha \in S \), if \( (\Sigma - \{ \alpha \}) \models \alpha \), then \( (\Sigma - \{ \alpha \}) \) is equivalent to \( \Sigma \). First, for any \( \Sigma \subseteq S \) we define \( Mod(\Sigma) = \{ h \in H \mid h \models \Sigma \} \).

**Lemma 1** For every \( \Sigma, \Gamma \subseteq S \), if \( Mod(\Sigma) = Mod(\Gamma) \), then \( Cn(\Sigma) = Cn(\Gamma) \).

We suggested proving the converse of the foregoing lemma as an exercise (hint: apply the Compactness Theorem).

**Exercise 1** For every \( \Sigma, \Gamma \subseteq S \), if \( Cn(\Sigma) = Cn(\Gamma) \), then \( Mod(\Sigma) = Mod(\Gamma) \).

1.2.10(b) Let \( \varphi_n = (A_0 \land \ldots \land A_n) \). Let \( \Sigma = \{ \varphi_n \mid n \in N \} \). Note that if \( \Gamma \subseteq \Sigma \) and \( \Gamma \) is independent, then \( \Gamma \) contains at most one sentence, whereas if \( \Gamma \) is equivalent to \( \Sigma \) then \( \Gamma \) is infinite. It follows that no subset of \( \Sigma \) is both independent and equivalent to \( \Sigma \).
1.2.10(c) Let $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$. We may suppose without loss of generality that no finite subset of $\Sigma$ is equivalent to $\Sigma$. We define a sequence of sentences $\delta_i \in \Sigma$ by induction as follows. Let $\delta_0 = \sigma_i$ where $i$ is the least $j$ such that $\sigma_j$ is not a tautology. Some such $j$ exists, for otherwise $\Sigma$ would be equivalent to $\emptyset$. Let $\delta_{n+1} = \sigma_i$ where $i$ is the least $j$ such that $\{\delta_0, \ldots, \delta_n\} \not\models \sigma_j$. Such a $j$ exists, for otherwise $\Sigma$ is equivalent to $\{\delta_0, \ldots, \delta_n\}$. Now, let $\gamma_0 = \delta_0$ and $\gamma_{n+1} = (\delta_0 \land \ldots \land \delta_n) \rightarrow \delta_{n+1}$. Let $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$. It is easy to verify that $\Gamma$ is equivalent to $\Sigma$ and that $\Gamma$ is independent.

We began to study the expressive power of sentential logic. We first addressed the finite case.

1. $O_n = \{A_i \mid i \leq n\}$;

2. $S_n =$ the set of sentences generated from $O_n$ using the sentential connectives;

3. $H_n = \{h \mid h : O_n \rightarrow \{\top, \bot\}\}$;

4. for $\varphi \in S_n$, $\text{Mod}_n(\varphi) = \{h \in H_n \mid h \models \varphi\}$.

**Theorem 1 (Expressive Completeness Theorem for Sentential Logic)**

For every $n$ and for every $X \subseteq H_n$, there is a $\varphi \in S_n$, such that $\text{Mod}_n(\varphi) = X$.

A proof of this theorem, essentially the same as we presented in class (with slightly different terminology), may be found in Enderton, section 1.5.

We then began to consider the infinite case. We showed that

**Theorem 2 (Cantor’s Diagonal Theorem)** $\mathcal{H}$ is not countable.

Proof: Let $\{h_1, h_2, \ldots\} \subseteq \mathcal{H}$. We show that there is an $h \in \mathcal{H}$ such that for every $i$, $h \neq h_i$. Let $\text{change}(\top) = \bot$ and $\text{change}(\bot) = \top$. For every $i$, let $h(A_i) = \text{change}(h_i(A_i))$. $\blacksquare$

**Exercise 2** Show that there is a $P \subseteq \mathcal{H}$ such that for all $\Sigma \subseteq S$, $\text{Mod}(\Sigma) \neq P$.

**Exercise 3** Show that for every finite $P \subseteq \mathcal{H}$ there is a $\Sigma \subseteq S$ such that $\text{Mod}(\Sigma) = P$.

Next time, we will go over solutions to these exercises, and perhaps Exercise 1 as well. We will then proceed to prove the compactness theorem for sentential logic (see Enderton, section 1.7). As part of the proof, we will present solutions to Enderton, exercises 1.7.1 & 2.