Math 603, Spring 2003, HW 5, due 3/31/2003

Part A

AIII) Write $A$ for an integral domain, $K = \text{Frac} \ A$, and set $\tilde{A} = \text{Int}_K(A)$. The domain $\tilde{A}$ is called the \textit{normalization} of $A$. Now set
\[ \mathcal{F} = (\tilde{A} \to A) = \{ \xi \in A \mid \xi \tilde{A} \subseteq A \} . \]

Of course, $\mathcal{F}$ is an ideal of $A$, called the \textit{conductor} of $A$ in $\tilde{A}$ (German: Führer). Check that $\mathcal{F}$ is also an ideal of $\tilde{A}$.

(a) If $S$ is a multiplicative subset in $A$, show $S^{-1}\tilde{A} = \text{Int}_K(S^{-1}A)$. Prove further, $S^{-1}A$ is normal if $\mathcal{F} \cap S \neq \emptyset$.

(b) Assume $\tilde{A}$ is a finitely generated $A$-module (frequently the case). Show that the conductor of $S^{-1}A$ in $S^{-1}\tilde{A}$ is the extended ideal $\mathcal{F}^e$. Show also in this case $S^{-1}A$ is normal if and only if $\mathcal{F} \cap S \neq \emptyset$.

(c) If $\tilde{A}$ is a finitely generated $A$-module, then
\[ \{ p \in \text{Spec} \ A \mid A_p \text{ is normal} \} \]

is open in Spec $A$, in fact it is a dense open of Spec $A$.

AII) A \textit{discrete valuation}, $\nu$, on a (commutative) ring $A$, is a function $\nu : A \to \mathbb{Z} \cup \{ \infty \}$ satisfying
\begin{itemize}
  \item[(a)] $\nu(xy) = \nu(x) + \nu(y)$
  \item[(b)] $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$
  \item[(c)] $\nu(x) = \infty \iff x = 0$.
\end{itemize}

A pair $(A, \nu)$ where $A$ a commutative ring and $\nu$ is a discrete valuation is called a \textit{discrete valuation ring} (DVR). Prove the following are equivalent:

(a) $A$ is a DVR

(b) $A$ is a local PID

(c) $A$ is a local, noetherian, normal domain of Krull dimension 1

(d) $A$ is a local, noetherian, normal domain and $(m_A \to a)( = \{ \xi \in \text{Frac} A \mid \xi m_A \subseteq A \}) \neq A$. Here, $m_A$ is the maximal ideal of $A$.

AIII) Let $A$ be a commutative ring with unity and assume $A$ is semi-local (it possesses just finitely many maximal ideals). Write $\mathcal{J}$ for the Jacobson radical of $A$ and give $A$ its $\mathcal{J}$-adic topology.

(a) Prove that $A$ is noetherian iff each maximal ideal of $A$ is finitely generated and each ideal is closed in the $\mathcal{J}$-adic topology.

(b) Assume $A$ is noetherian, then the map $A \to A_{\text{rad}}$ gives $A_{\text{rad}}$ its $\mathcal{J}$-adic topology. If $A_{\text{rad}}$ is complete prove that $A$ is complete.

AIV) (a) Let $A$ be a local ring, give $A$ its $m$-adic topology ($m = m_A$ is the maximal ideal of $A$) and assume $A$ is complete. Given an $A$-algebra, $B$, suppose $B$ is finitely generated as an $A$-module. Prove that $B$ is a finite product of $A$-algebras each of which is a local ring. Give an example to show that some hypothesis like completeness is necessary for the conclusion to be valid.

(b) Again $A$ is complete and local, assume $f(X) \in A[X]$ is a monic polynomial. Write $\overline{f}(X)$ for the image of $f$ in $(A/m)[X]$. If $\overline{f}(X) \mid \overline{f}(X) = \overline{g}(X)\overline{h}(X)$ where $g$ and $h$ are relatively prime in $(A/m)[X]$, show that $f$ factors as $G(X)H(X)$ where $G(X) = g(x); \overline{f}(X) = \overline{h}(X)$. What can you say about $\deg G$, $\deg H$ and uniqueness of this factorization? Compare parts (a) and (b).
Part B

BII) In this problem, $A$ is an integral domain and $k = \text{Frac } A$. If $\nu$ and $\omega$ are two discrete valuations of $k$ (c.f. AII, the functions $\nu$ and $\omega$ are defined on $A$ and extended to $k$ via $\nu(a/b) = \nu(a) - \nu(b)$, etc.), let's call $\nu$, $\omega$ inequivalent iff one is not a constant multiple of the other. Write $S$ for a set of inequivalent discrete valuations of $k$ and say that $A$ is adapted to $S$ provided

$$A = \{ x \in k \mid (\forall \nu \in S)(\nu(x) \geq 0) \}.$$ 

(a) Prove the following are equivalent:
   i. $A$ is a Dedekind domain
   ii. $(\forall$ ideals, $a$, of $A)(\forall x, x \neq 0, x \in a)(\exists y \in a)(a = (x, y))$.
   iii. There is a family of discrete valuations of $k$, say $S$, for which $A$ is adapted to $S$ and so that the following holds:
   $$(\forall \nu, \omega \in S) (\nu \neq \omega \implies (\exists a \in A)(\nu(a) \geq 1 \text{ and } \omega(a - 1) \geq 1)).$$

(b) Vis a vis part (a), describe a one-to-one correspondence $S \leftrightarrow \text{Max}(A)$.

(c) Take $k = \mathbb{Q}$, consider all prime numbers $p$ with $p \equiv 1 \pmod{4}$, write $\text{ord}_p(n)$ for the highest exponent, $e$, so that $p^e \mid n$. Then $\text{ord}_p$ is a discrete valuation of $\mathbb{Q}$, and we set $S = \{ \text{ord}_p \mid p \equiv 1 \pmod{4} \}$. Illustrate iii in part (a) above with this $S$. What is $A$, in concrete terms? It is pretty clear now how to make many Dedekind domains.

(d) Say $A$ is a Dedekind domain and $a, b$ are two non-zero ideals of $A$. Show $\exists x \in k(= \text{Frac } A)$, so that $a + xb = A$.

(e) Again let $A$ be a Dedekind domain and let $L$ be a finite subset of $\text{Max}(A)$. Write $A^L = \bigcap\{A_p \mid p \not\in L\}$, then $A \subseteq A^L$ and so $\mathbb{G}_m(A) \subseteq \mathbb{G}_m(A^L)$. Recall, $\mathbb{G}_m(B)$ is the group of units of the ring $B$. Prove that Pic($A$) is a torsion group $\iff \mathbb{G}_m(A^L)/\mathbb{G}_m(A)$ is a free abelian group of rank $\#(L)$ for every finite set $L$ of $\text{Max}(A)$.

BII) Here, $k$ is a field and $A = k[X_\alpha]_{\alpha \in I}$. The index set, $I$, may possibly be infinite. Write $m$ for the fractional ideal generated by all the $X_\alpha$, $\alpha \in I$. Set $A_i = A/m^{i+1}$, so $A_0 = k$. These $A_i$ form a left mapping system and we set

$$\hat{A} = \lim_{\leftarrow} A_i$$

and call $\hat{A}$ the completion of $A$ in the $m$-adic topology. Note that the kernel of $\hat{A} \to A_j$ is the closure of $m^{j+1}$ in $\hat{A}$.

(a) Show that $\hat{A}$ is canonically isomorphic to the ring of formal power series in the $X_\alpha$ in which only finitely many monomials of each degree occur.

(b) Now let $I = \mathbb{N}$ (the counting numbers) and write $\hat{m}$ for the closure of $m$ in $\hat{A}$. By adapting Cantor’s diagonal argument, prove that $\hat{m}$ is NOT $\hat{A}m$. Which is bigger?

(c) Again, $I$ as in (b). Let $k$ be a finite field, prove the Lemma. If $k$ is a finite field and $\lambda > 0$, $(\exists n_\lambda)(\forall n \geq n_\lambda)$, $\exists a \text{ homogeneous polynomial, } F_n \in k[n^2 \text{ variables}], \text{ so that } \deg F_n = n$ and $F_n \text{ cannot be written as the sum of terms of degree } n \text{ of any polynomial } P_1Q_1 + \cdots + P_nQ_n$, where $P_j$, $Q_j$ are in $k[n^2 \text{ variables}]$ and have no constant term.

    Use the lemma to prove $(\hat{m})^2 \neq (\hat{m}^2)$.

(d) Use (b) and (c) to prove that $\hat{A}$ is NOT complete in the $\hat{m}$-adic topology.

(e) All the pathology exhibited in (b), (c) and (d) arises as $I$ is not finite, indeed when $I$ is finite, prove:
i. \( \hat{m} \) is A\( \hat{m} \);
ii. \( \hat{m}^2 = (m^2) \);
iii. \( \hat{A} \) is complete in the \( \hat{m} \)-adic topology.

BIII) Say \( X \) denotes the category \( \text{TOP} \) (topological spaces and continuous maps) and \( \text{Haus}(X) \) the full subcategory of Hausdorff topological spaces.

(a) At first, use the ordinary Cartesian product in \( X \), with the product topology. Denote this \( Y \times Z \).
Show that \( Y \in \text{Haus}(X) \iff \) the diagonal map \( \Delta : Y \to Y \times Y \) is closed.

(b) For \( X, Y \in \text{Haus}(X) \), recall that \( X \xrightarrow{f} Y \) is called a proper map \( \iff f^{-1}(\text{compact}) = \text{compact} \).
(Of course, any map \( f : X \to Y \) will be proper if \( X \) is compact.) Show that \( f : X \to Y \) is proper iff \( (\forall T \in \text{Haus}(X))(f_T : X \times T \to Y \times T \) is a closed map.)

(c) With (a) and (b) as background look at another subcategory, \( X_A \) of \( X \): here \( A \) is a commutative ring, \( X_A \) consists of the topological spaces \( \text{Spec} B \), where \( B \) is an \( A \)-algebra. Maps in \( X_A \) are those coming from homomorphisms of \( A \)-algebras, viz: \( B \to C \) gives \( \text{Spec} C \to \text{Spec} B \).
Define
\[
(\text{Spec} B) \amalg (\text{Spec} C) = \text{Spec} (B \otimes_A C)
\]
and prove that \( X_A \) possesses products.

NB:

i. The topology on \( \text{Spec} B \amalg \text{Spec} C \) is NOT the product topology—it is stronger (more opens and closeds)
ii. \( \text{Spec} B \amalg \text{Spec} C \neq \text{Spec} B \times \text{Spec} C \) as sets.

Prove: the diagonal map \( \Delta_Y : Y \to Y \amalg Y \) is closed \( (Y = \text{Spec} B) \). This recaptures (a) in the non-Hausdorff setting of \( X_A \).

(d) Given \( f : \text{Spec} C \to \text{Spec} B \) (arising from an \( A \)-algebra map \( B \to C \)) call \( f \) proper \( \iff i) C \) is a finitely generated \( B \)-algebra and ii) \( (\forall T = \text{Spec} D)(f_T : \text{Spec} C \amalg \text{Spec} D \to \text{Spec} B \amalg \text{Spec} D \) is a closed map.)

Prove: if \( C \) is integral over \( B \), then \( f \) is proper. However, prove also, \( \text{Spec} (B[T]) \to \text{Spec} B \) is never proper.

(e) Say \( A = C \). For which \( A \)-algebras \( B \) is the map \( \text{Spec} B \to \text{Spec} A \) proper?

BIV) \( A \) is noetherian local, \( m_A \) its maximal ideal, and
\[
\hat{A} = \lim_{\to \infty} A/m_A^{n+1} = \text{completion of} \ A \ in \ the \ m \text{-adic topology}.
\]

Let \( B, m_B \) be another noetherian local ring and its maximal ideal. Assume \( f : A \to B \) is a ring homomorphism and we always assume \( f(m_A) \subseteq m_B \).

(a) Prove: \( f \) gives rise to a homomorphism \( \hat{A} \xrightarrow{\hat{f}} \hat{B} \) (and \( \hat{m}_A \to \hat{m}_B \)).

(b) Prove: \( \hat{f} \) is an isomorphism \( \iff \)

i. \( B \) is flat over \( A \)
ii. \( f(m_A) \cdot B = m_B \)
iii. \( A/m_A \to B/m_B \) is an isomorphism.