Math 602, Fall 2002, HW 4, due 11/18/2002

Part A

AI) Recall that for every integral domain, $A$, there is a field, Frac($A$), containing $A$ minimal among all fields containing $A$. If $B$ is an $A$-algebra, an element $b \in B$ is integral over $A$ $\iff$ there exists a monic polynomial, $f(X) \in A[X]$, so that $f(b) = 0$. The domain, $A$, is integrally closed in $B$ if every $b \in B$ which is integral over $A$ actually comes from $A$ (via the map $A \to B$). The domain, $A$, is integrally closed (also called normal) if it is integrally closed in Frac($A$).

(a) $A$ is integrally closed $\iff$ $A[X]/(f(X))$ is an integral domain for every MONIC irreducible polynomial, $f(X)$.

(b) $A$ is a UFD $\iff A$ possesses the ACC on principal ideals and $A[X]/(f(X))$ is an integral domain for every irreducible polynomial $f(X)$. (It follows that every UFD is a normal domain.)

(c) If $k$ is a field and characteristic of $k$ is not 2, show that $A = k[X,Y,Z,W]/(XY-ZW)$ is a normal domain. What happens if char($k$) = 2?

AII) If $A$ is a ring, write End$^*(A)$ for the collection of surjective ring endomorphisms of $A$. Suppose $A$ is commutative and noetherian, prove End$^*(A) = Aut(A)$.

AIII) Write $M(n,A)$ for the ring of all $n \times n$ matrices with entries from $A$ ($A$ is a ring). Suppose $K$ and $k$ are fields and $K \supseteq k$.

(a) Show that if $M,N \in M(n,k)$ and if $\exists P \in GL(n,K)$ so that $PMP^{-1} = N$, then $\exists Q \in GL(n,k)$ so that $QM Q^{-1} = N$.

(b) Prove that (a) is false for rings $B \supseteq A$ via the following counterexample: $A = \mathbb{R}[X,Y]/(X^2 + Y^2 - 1)$, $B = \mathbb{C}[X,Y]/(X^2 + Y^2 - 1)$. Find two matrices similar in $M(2,B)$ but NOT similar in $M(2,A)$.

(c) Let $S^n$ be the $n$-sphere and represent $S^n \subseteq \mathbb{R}^{n+1}$ as $\{(z_0, \ldots, z_n) \in \mathbb{R}^{n+1} | \sum_{j=0}^{n} z_j^2 = 1\}$. Show that there is a natural injection of $\mathbb{R}[X_0, \ldots, X_n]/(\sum_{j=0}^{n} X_j^2 - 1)$ into $C(S^n)$, the ring of (real valued) continuous functions on $S^n$. Prove further that the former ring is an integral domain but $C(S^n)$ is not. Find the group of units in the former ring.

AIV) (Rudakov) Say $A$ is a ring and $M$ is a rank 3 free $A$-module. Write $Q$ for the bilinear form whose matrix (choose some basis for $M$) is

$$
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1 
\end{pmatrix}.
$$

Thus, if $v = (x,y,z)$ and $w = (\xi, \eta, \zeta)$, we have

$$
Q(v, w) = (x,y,z) \begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1 
\end{pmatrix} \begin{pmatrix}
\xi \\
\eta \\
\zeta 
\end{pmatrix}.
$$

Prove that $Q(w, v) = Q(v, Bw)$ with $B = I + \text{nilpotent} \iff a^2 + b^2 + c^2 = abc$.

AV) Let $M$ be a $\Lambda$-module ($\Lambda$ is not necessarily commutative) and say $N$ and $N'$ are submodules of $M$.

(a) Suppose $N + N'$ and $N \cap N'$ are f.g. $\Lambda$-modules. Prove that both $N$ and $N'$ are then f.g. $\Lambda$-modules.

(b) Give a generalization to finitely many submodules, $N_1, \ldots, N_t$ of $M$.  

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(c) Can you push part (b) to an infinite number of $N_j$?

(d) If $M$ is noetherian as a $\Lambda$-module, is $\Lambda$ necessarily noetherian as a ring (left noetherian as $M$ is a left module)? What about $\mathfrak{X} = \Lambda/\text{Ann}(M)$?

Part B

BI) (Continuation of AI)

(a) Consider the ring $A(n) = \mathbb{C}[X_1, \ldots, X_n]/(X_1^2 + \cdots + X_n^2)$. There is a condition on $n$, call it $C(n)$, so that $A(n)$ is a UFD iff $C(n)$ holds. Find explicitly $C(n)$ and prove that theorem.

(b) Consider the ring $B(n) = \mathbb{C}[X_1, \ldots, X_n]/(X_1^2 + X_2^2 + X_3^2 + \cdots + X_n^2)$. There is a condition on $n$, call it $D(n)$, so that $B(n)$ is a UFD iff $D(n)$ holds. Find explicitly $D(n)$ and prove the theorem.

(c) Investigate exactly what you can say if $C(n)$ (respectively $D(n)$) does not hold.

(d) Replace $\mathbb{C}$ by $\mathbb{R}$ and answer (a) and (b).

(e) Can you formulate a theorem about the ring $A[X, Y]/(f(X, Y))$, where $A$ is a given UFD and $f$ is a polynomial in $A[X, Y]$, of the form $A[X, Y]/(f(X, Y))$ is a UFD provided $f(X, Y) \cdots$? Your theorem must be general enough to yield (a) and (b) as easy consequences. (You must prove it too.)

BII) (Exercise on projective modules) In this problem, $A \in \text{Ob}(CR)$.

(a) Suppose $P$ and $P'$ are projective $A$-modules, and $M$ is an $A$-module. If

$$0 \to K \to P \to M \to 0 \quad \text{and} \quad 0 \to K' \to P' \to M \to 0$$

are exact, prove that $K' \oplus P \cong K \oplus P'$.

(b) If $P$ is a f.g. projective $A$-module, write $P^D$ for the $A$-module $\text{Hom}_A(P, A)$. We have a canonical map $P \to P^{DD}$. Prove this is an isomorphism.

(c) Again, $P$ is f.g. projective; suppose we’ve given an $A$-linear map $\mu : \text{End}_A(P) \to A$. Prove: there exists a unique element $f \in \text{End}_A(P)$ so that $(\forall h \in \text{End}_A(P))(\mu(h) = \text{tr}(hf))$. Here, you must define the trace, $\text{tr}$, for f.g. projectives, $P$, as a well-defined map, then prove the result.

(d) Again, $P$ is f.g. projective; $\mu$ is as in (c). Show that $\mu(gh) = \mu(hg) \iff \mu = a \text{ tr}$ for some $a \in A$.

(e) Situation as in (b), then each $f \in \text{End}_A(P)$ gives rise to $f^D \in \text{End}_A(P^{DD})$. Show that $\text{tr}(f) = \text{tr}(f^D)$.

(f) Using categorical principles, reformulate (a) for injective modules and prove your reformulation.

BIII) Suppose $K$ is a commutative ring and $a, b \in K$. Write $A = K[T]/(T^2 - a)$; there is an automorphism of $A$ (the identity on $K$) which sends $t$ to $-t$, where $t$ is the image of $T$ in $A$. If $\xi, \eta \in A$, we write $\overline{\xi}$ for the image of $\xi$ under this automorphism. Let $\mathbb{H}(K; a, b)$ denote the set

$$\mathbb{H}(K; a, b) = \left\{ \begin{pmatrix} \xi & b\eta \\ \eta & -\xi \end{pmatrix} \middle| \xi, \eta \in A \right\},$$

this is a subring of the $2 \times 2$ matrices over $A$. Observe that $q \in \mathbb{H}(K; a, b)$ is a unit there iff $q$ is a unit of the $2 \times 2$ matrices over $A$.

(a) Consider the non-commutative polynomial ring $K\langle X, Y \rangle$. There is a 2-sided ideal, $\mathcal{I}$, in $K\langle X, Y \rangle$ so that $\mathcal{I}$ is symmetrically generated vis a vis $a$ and $b$ and $K\langle X, Y \rangle/\mathcal{I}$ is naturally isomorphic to $\mathbb{H}(K; a, b)$. Find the generators of $\mathcal{I}$ and establish the explicit isomorphism.
(b) For pairs \((a, b)\) and \((\alpha, \beta)\) decide exactly when \(\mathbb{H}(K; a, b)\) is isomorphic to \(\mathbb{H}(K; \alpha, \beta)\) as objects of the comma category \(\text{RNG}^K\).

(c) Find all isomorphism classes of \(\mathbb{H}(K; a, b)\) when \(K = \mathbb{R}\) and when \(K = \mathbb{C}\). If \(K = \mathbb{F}_p, p \neq 2\) answer the same question and then so do for \(\mathbb{F}_2\).

(d) When \(K\) is just some field, show \(\mathbb{H}(K; a, b)\) is a “division ring” (all non-zero elements are units) \(\iff\) the equation \(X^2 - aY^2 = b\) has no solution in \(K\) (here we assume \(a\) is not a square in \(K\)). What is the case if \(a\) is a square in \(K\)?

(e) What is the center of \(\mathbb{H}(K; a, b)\)?

(f) For the field \(K = \mathbb{Q}\), prove that \(\mathbb{H}(\mathbb{Q}; a, b)\) is a division ring \(\iff\) the surface \(aX^2 + bY^2 = Z^2\) has no points whose coordinates are integers except 0.

\[BVI\] (a) If \(A\) is a commutative ring and \(f(X) \in A[X]\), suppose \((\exists g(X) \neq 0)(g(X) \in A[X] \text{ and } g(X)f(X) = 0)\). Show: \((\exists \alpha \in A)(\alpha \neq 0 \text{ and } \alpha f(X) = 0)\). \textit{Caution:} \(A\) may possess non-trivial nilpotent elements.

(b) Say \(K\) is a field and \(A = K[X_{ij}, 1 \leq i, j \leq n]\). The matrix

\[
M = \begin{pmatrix}
X_{11} & \cdots & X_{1n} \\
\vdots & \ddots & \vdots \\
X_{n1} & \cdots & X_{nn}
\end{pmatrix}
\]

has entries in \(A\) and \(\det(M) \in A\). Prove that \(\det(M)\) is an irreducible polynomial of \(A\).

\[BV\] Let \(A\) be a commutative noetherian ring and suppose \(B\) is a commutative \(A\)-algebra which is f.g. as an \(A\)-algebra. If \(G \subseteq \text{Aut}_{A\text{-alg}}(B)\) is a finite subgroup, write

\[
B^G = \{b \in B \mid \sigma(b) = b, \text{ all } \sigma \in G\}.
\]

Prove that \(B^G\) is also f.g. as an \(A\)-algebra; hence \(B^G\) is noetherian.

\[BVI\] Again, \(A\) is a commutative ring. Write \(\text{RCF}(A)\) for the ring of \(\infty \times \infty\) matrices all of whose rows and all of whose columns possess but finitely many \(\textit{not} \text{ bounded}\) non-zero entries. This \(\textit{is}\) a ring under ordinary matrix multiplication \(\text{(as you see easily)}\).

(a) Specialize to the case \(A = \mathbb{C}\); find a \textit{maximal} two-sided ideal, \(\mathcal{E}\), of \(\text{RCF}(\mathbb{C})\). Prove it is such and is the only such. You are to find \(\mathcal{E}\) explicitly. Write \(E(\mathbb{C})\) for the ring \(\text{RCF}(\mathbb{C})/\mathcal{E}\).

(b) Show that there exists a natural injection of rings \(M_n = n \times n\) complex matrices \(\hookrightarrow \text{RCF}(\mathbb{C})\) so that the composition \(M_n \to E(\mathbb{C})\) is still injective; show further that if \(p \mid q\) we have a commutative diagram

\[
\begin{array}{ccc}
M_p & \hookrightarrow & M_q \\
\downarrow & & \downarrow \\
E(\mathbb{C}) & & E(\mathbb{C})
\end{array}
\]

\[BVII\] \text{(Left and right noetherian)} For parts (a) and (b), let \(A = \mathbb{Z}(X,Y)/(XY,Y^2)\)—a non-commutative ring.

(a) Prove that \(\mathbb{Z}[X] \hookrightarrow \mathbb{Z}[X,Y] \to A\) is an injection and that \(A = \mathbb{Z}[X] \amalg (\mathbb{Z}[X]y)\) as a left \(\mathbb{Z}[X]\)-module \((y\text{ is the image of } Y \text{ in } A)\); hence \(A\) is a left noetherian ring.

(b) However, the right ideal generated by \(\{X^ny \mid n \geq 0\}\) is NOT f.g. \(\text{(prove!)}\); so, \(A\) is not right noetherian.
(c) Another example. Let
\[ C = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in \mathbb{Z}; b, c, \in \mathbb{Q} \right\}. \]

Then $C$ is right noetherian but NOT left noetherian (prove!).