Problem B1(60 pts). (1) Implement the Gram-Schmidt orthonormalization procedure and the modified Gram-Schmidt procedure. You may use the pseudo-code showed in (2).

(2) Implement the method to compute the QR decomposition of an invertible matrix. You may use the following pseudo-code:

```
function qr(A: matrix): [Q, R] pair of matrices
    begin
        n = dim(A);
        R = 0; (the zero matrix)
        Q1(:, 1) = A(:, 1);
        R(1, 1) = sqrt(Q1(:, 1)\(\top\) \cdot Q1(:, 1));
        Q(:, 1) = Q1(:, 1)/R(1, 1);
        for k := 1 to n – 1 do
            w = A(:, k + 1);
            for i := 1 to k do
                R(i, k + 1) = A(:, k + 1)\(\top\) \cdot Q(:, i);
                w = w – R(i, k + 1)Q(:, i);
            endfor;
            Q1(:, k + 1) = w;
            R(k + 1, k + 1) = sqrt(Q1(:, k + 1)\(\top\) \cdot Q1(:, k + 1));
            Q(:, k + 1) = Q1(:, k + 1)/R(k + 1, k + 1);
        endfor;
    end;
```

Test it on various matrices.

(3) Given any invertible matrix A, define the sequences $A_k$, $Q_k$, $R_k$ as follows:

\[
A_1 = A \\
Q_k R_k = A_k \\
A_{k+1} = R_k Q_k
\]
for all $k \geq 1$, where in the second equation, $Q_k R_k$ is the QR decomposition of $A_k$ given by part (2).

Run the above procedure for various values of $k$ (50, 100, ...) and various real matrices $A$, in particular some symmetric matrices; also run the Matlab command `eig` on $A_k$, and compare the diagonal entries of $A_k$ with the eigenvalues given by $\text{eig}(A_k)$.

What do you observe? How do you explain this behavior?

Problem B2(40 pts). Refer to Problem B4 of HW6. Write a Matlab program to compute a logarithm $B$ of a rotation matrix $R \in \text{SO}(3)$, namely a skew-symmetric matrix $B$ such that $e^B = R$, using the method described in Problem B4. In particular, compute a log $B$ of $R$ such that the angle $\theta$ associated with $B$ satisfies the condition $0 \leq \theta \leq \pi$. If $\text{tr}(R) \neq -1$, then $\theta < \pi$ and the matrix $B$ associated with $\theta$ is uniquely determined. Otherwise $\theta = \pi$ and $B$ is determined up to sign.

Problem B3(210 pts). Refer to Problem B5 of HW6. Similitudes can be used to describe certain deformations (or flows) of a deformable body $B_t$ in 3D. Given some initial shape $B$ in $\mathbb{R}^3$ (for example, a sphere, a cube, etc.), a deformation of $B$ is given by a piecewise differentiable curve

$$D: [0, T] \rightarrow \text{SIM}(3),$$

where each $D(t)$ is a similitude (for some $T > 0$). The deformed body $B_t$ at time $t$ is given by

$$B_t = D(t)(B),$$

where $D(t) \in \text{SIM}(3)$ is a similitude.

The surjectivity of the exponential map $\exp: \text{sim}(3) \rightarrow \text{SIM}(3)$ implies that there is a map $\log: \text{SIM}(3) \rightarrow \text{sim}(3)$, although it is multivalued. The exponential map and the log “function” allows us to work in the simpler (noncurved) Euclidean space $\text{sim}(3)$ (which has dimension 7).

For instance, given two similitudes $A_1, A_2 \in \text{SIM}(3)$ specifying the shape of $B$ at two different times, we can compute $\log(A_1)$ and $\log(A_2)$, which are just elements of the Euclidean space $\text{sim}(3)$, form the linear interpolant $(1 - t)\log(A_1) + t \log(A_2)$, and then apply the exponential map to get an interpolating deformation

$$t \mapsto e^{(1-t)\log(A_1)+t\log(A_2)}, \quad t \in [0, 1].$$

Also, given a sequence of “snapshots” of the deformable body $B$, say $A_0, A_1, \ldots, A_m$, where each is $A_i$ is a similitude, we can try to find an interpolating deformation (a curve in $\text{SIM}(3)$) by finding a simpler curve $t \mapsto C(t)$ in $\text{sim}(3)$ (say, a $B$-spline) interpolating $\log A_0, \log A_1, \ldots, \log A_m$. Then, the curve $t \mapsto e^{C(t)}$ yields a deformation in $\text{SIM}(3)$ interpolating $A_0, A_1, \ldots, A_m$.

(1) (60 pts). Write a program interpolating between two deformations, using the formulae found in Problems B4 and B5 of HW6 (not the built-in Matlab functions!).
(2) (150 pts). Write a program using your cubic spline interpolation program from the first project, to interpolate a sequence of deformations given by similitudes $A_0, A_1, \ldots, A_m$ in $\text{SIM}(3)$. Use the formulae found in Problems B4 and B5 of HW6 (not the built-in \texttt{Matlab} functions!).

\textbf{TOTAL: 410 points.}