Fall, 2014  CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 6

November 28, 2014; Due December 16, 2014
Beginning of class

Problem B1 (20 +∞2/∞1 ≈ 40 pts). (1) Let $H$ be the affine hyperplane in $\mathbb{R}^n$ given by the equation

$$a_1 x_1 + \cdots + a_n x_n = c,$$

with $a_i \neq 0$ for some $i, 1 \leq i \leq n$. The linear hyperplane $H_0$ parallel to $H$ is given by the equation

$$a_1 x_1 + \cdots + a_n x_n = 0,$$

and we say that a vector $y \in \mathbb{R}^n$ is orthogonal (or perpendicular) to $H$ iff $y$ is orthogonal to $H_0$. Let $h$ be the intersection of $H$ with the line through the origin and perpendicular to $H$. Prove that the coordinates of $h$ are given by

$$c \frac{1}{a_1^2 + \cdots + a_n^2} (a_1, \ldots, a_n).$$

(2) For any point $p \in H$, prove that $\|h\| \leq \|p\|$. Thus, it is natural to define the distance $d(O, H)$ from the origin $O$ to the hyperplane $H$ as $d(O, H) = \|h\|$. Prove that

$$d(O, H) = \frac{|c|}{(a_1^2 + \cdots + a_n^2)^\frac{1}{2}}.$$

(3) Let $S$ be a finite set of $n \geq 3$ points in the plane ($\mathbb{R}^2$). Prove that if for every pair of distinct points $p_i, p_j \in S$, there is a third point $p_k \in S$ (distinct from $p_i$ and $p_j$) such that $p_i, p_j, p_k$ belong to the same (affine) line, then all points in $S$ belong to a common (affine) line.

*Hint.* Proceed by contradiction and use a minimality argument. This is either $\infty$-hard or relatively easy, depending how you proceed!

Problem B2 (10 pts). Let $A$ be any real or complex $n \times n$ matrix and let $\|\|$ be any operator norm.
Prove that for every $m \geq 1$,
\[
\|I\| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e\|A\|.
\]

If you know some analysis, deduce from the above that the sequence $(E_m)$ of matrices
\[
E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}
\]
converges to a limit denoted $e^A$, and called the exponential of $A$.

**Problem B3 (90 pts).** (The space of closed polygons in $\mathbb{R}^2$, after Hausmann and Knutson)

An open polygon $P$ in the plane is a sequence $P = (v_1, \ldots, v_{n+1})$ of point $v_i \in \mathbb{R}^2$ called vertices (with $n \geq 1$). A closed polygon, for short a polygon, is an open polygon $P = (v_1, \ldots, v_{n+1})$ such that $v_{n+1} = v_1$. The sequence of edge vectors $(e_1, \ldots, e_n)$ associated with the open (or closed) polygon $P = (v_1, \ldots, v_{n+1})$ is defined by
\[
e_i = v_{i+1} - v_i, \quad i = 1, \ldots, n.
\]
Thus, a closed or open polygon is also defined by a pair $(v_1, (e_1, \ldots, e_n))$, with the vertices given by
\[
v_{i+1} = v_i + e_i, \quad i = 1, \ldots, n.
\]
Observe that a polygon $(v_1, (e_1, \ldots, e_n))$ is closed iff
\[
e_1 + \cdots + e_n = 0.
\]
Since every polygon $(v_1, (e_1, \ldots, e_n))$ can be translated by $-v_1$, so that $v_1 = (0, 0)$, we may assume that our polygons are specified by a sequence of edge vectors.

Recall that the plane $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$, via the isomorphism
\[
(x, y) \mapsto x + iy.
\]
We will represent each edge vector $e_k$ by the square of a complex number $w_k = a_k + ib_k$. Thus, every sequence of complex numbers $(w_1, \ldots, w_n)$ defines a polygon (namely, $(w_1^2, \ldots, w_n^2)$). This representation is many-to-one: the sequences $(\pm w_1, \ldots, \pm w_n)$ describe the same polygon. To every sequence of complex numbers $(w_1, \ldots, w_n)$, we associate the pair of vectors $(a, b)$, with $a, b \in \mathbb{R}^n$, such that if $w_k = a_k + ib_k$, then
\[
a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n).
\]
The mapping
\[
(w_1, \ldots, w_n) \mapsto (a, b)
\]
is clearly a bijection, so we can also represent polygons by pairs of vectors \((a, b) \in \mathbb{R}^n \times \mathbb{R}^n\).

(a) Prove that a polygon \(P\) represented by a pair of vectors \((a, b) \in \mathbb{R}^n \times \mathbb{R}^n\) is closed iff \(a \cdot b = 0\) and \(\|a\|_2 = \|b\|_2\).

(b) Given a polygon \(P\) represented by a pair of vectors \((a, b) \in \mathbb{R}^n \times \mathbb{R}^n\), the length \(l(P)\) of the polygon \(P\) is defined by \(l(P) = |w_1|^2 + \cdots + |w_n|^2\), with \(w_k = a_k + ib_k\). Prove that
\[
l(P) = \|a\|_2^2 + \|b\|_2^2.
\]

Deduce from (a) and (b) that every closed polygon of length 2 with \(n\) edges is represented by a \(n \times 2\) matrix \(A\) such that \(A^\top A = I\).

Remark: The space of all \(n \times 2\) real matrices \(A\) such that \(A^\top A = I\) is a space known as the Stiefel manifold \(S(2, n)\).

(c) Recall that in \(\mathbb{R}^2\), the rotation of angle \(\theta\) specified by the matrix
\[
R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]
is expressed in terms of complex numbers by the map
\[z \mapsto ze^{i\theta}.
\]

Let \(P\) be a polygon represented by a pair of vectors \((a, b) \in \mathbb{R}^n \times \mathbb{R}^n\). Prove that the polygon \(R_\theta(P)\) obtained by applying the rotation \(R_\theta\) to every vertex \(w_k^2 = (a_k + ib_k)^2\) of \(P\) is specified by the pair of vectors
\[
(\cos(\theta/2)a - \sin(\theta/2)b, \sin(\theta/2)a + \cos(\theta/2)b) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.
\]

(d) The reflection \(\rho_x\) about the \(x\)-axis corresponds to the map
\[z \mapsto z,
\]
whose matrix is,
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Prove that the polygon \(\rho_x(P)\) obtained by applying the reflection \(\rho_x\) to every vertex \(w_k^2 = (a_k + ib_k)^2\) of \(P\) is specified by the pair of vectors
\[
(a, -b) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

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(e) Let \( Q \in \text{O}(2) \) be any isometry such that \( \det(Q) = -1 \) (a reflection). Prove that there is a rotation \( R_{-\theta} \in \text{SO}(2) \) such that

\[
Q = \rho_x \circ R_{-\theta}.
\]

Prove that the isometry \( Q \), which is given by the matrix

\[
Q = \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix},
\]

is the reflection about the line corresponding to the angle \( \theta/2 \) (the line of equation \( y = \tan(\theta/2)x \)).

Prove that the polygon \( Q(P) \) obtained by applying the reflection \( Q = \rho_x \circ R_{-\theta} \) to every vertex \( w^2 = (a_k + ib_k)^2 \) of \( P \), is specified by the pair of vectors

\[
\begin{pmatrix}
\cos(\theta/2)a + \sin(\theta/2)b, \\
\sin(\theta/2)a - \cos(\theta/2)b
\end{pmatrix} = \begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2 \\
\vdots & \vdots \\
a_n & b_n
\end{pmatrix} \begin{pmatrix}
\cos(\theta/2) & \sin(\theta/2) \\
\sin(\theta/2) & -\cos(\theta/2)
\end{pmatrix}.
\]

(f) Define an equivalence relation \( \sim \) on \( S(2, n) \) such that if \( A_1, A_2 \in S(2, n) \) are any \( n \times 2 \) matrices such that \( A_1^T A_1 = A_2^T A_2 = I \), then

\[
A_1 \sim A_2 \iff A_2 = A_1 Q \quad \text{for some } Q \in \text{O}(2).
\]

Prove that the quotient \( G(2, n) = S(2, n)/\sim \) is in bijection with the set of all 2-dimensional subspaces (the planes) of \( \mathbb{R}^n \). The space \( G(2, n) \) is called a Grassmannian manifold.

Prove that up to translations and isometries in \( \text{O}(2) \) (rotations and reflections), the \( n \)-sided closed polygons of length 2 are represented by planes in \( G(2, n) \).

**Problem B4 (100 pts).** (a) For any matrix

\[
A = \begin{pmatrix}
0 & -c & b \\
c & 0 & -a \\
-b & a & 0
\end{pmatrix},
\]

if we let \( \theta = \sqrt{a^2 + b^2 + c^2} \) and

\[
B = \begin{pmatrix}
a^2 & ab & ac \\
ab & b^2 & bc \\
ac & bc & c^2
\end{pmatrix},
\]

prove that

\[
A^2 = -\theta^2 I + B,
\]

\[
AB = BA = 0.
\]
From the above, deduce that
\[ A^3 = -\theta^2 A. \]

(b) Prove that the exponential map \( \exp: \mathfrak{so}(3) \to \text{SO}(3) \) is given by
\[ \exp A = e^A = \cos \theta I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} B, \]
or, equivalently, by
\[ e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2, \quad \text{if } \theta \neq 0, \]
with \( \exp(0) = I_3. \)

(c) Prove that \( e^A \) is an orthogonal matrix of determinant +1, i.e., a rotation matrix.

(d) Prove that the exponential map \( \exp: \mathfrak{so}(3) \to \text{SO}(3) \) is surjective. For this, proceed as follows: Pick any rotation matrix \( R \in \text{SO}(3); \)

(1) The case \( R = I \) is trivial.

(2) If \( R \neq I \) and \( \text{tr}(R) \neq -1, \) then
\[ \exp^{-1}(R) = \left\{ \theta \frac{2}{\sin \theta} (R - R^T) \bigg| 1 + 2 \cos \theta = \text{tr}(R) \right\}. \]
(Recall that \( \text{tr}(R) = r_{11} + r_{22} + r_{33}, \) the trace of the matrix \( R). \)

Show that there is a unique skew-symmetric \( B \) with corresponding \( \theta \) satisfying \( 0 < \theta < \pi \) such that \( e^B = R. \)

(3) If \( R \neq I \) and \( \text{tr}(R) = -1, \) then prove that the eigenvalues of \( R \) are \( 1, -1, -1, \) that \( R = R^\top, \) and that \( R^2 = I. \) Prove that the matrix
\[ S = \frac{1}{2}(R - I) \]
is a symmetric matrix whose eigenvalues are \( -1, -1, 0. \) Thus, \( S \) can be diagonalized with respect to an orthogonal matrix \( Q \) as
\[ S = Q \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} Q^\top. \]

Prove that there exists a skew symmetric matrix
\[ U = \begin{pmatrix}
0 & -d & c \\
d & 0 & -b \\
-c & b & 0
\end{pmatrix} \]
so that
\[ U^2 = S = \frac{1}{2} (R - I). \]

Observe that
\[
U^2 = \begin{pmatrix}
-(c^2 + d^2) & bc & bd \\
bc & -(b^2 + d^2) & cd \\
bd & cd & -(b^2 + c^2)
\end{pmatrix},
\]
and use this to conclude that if \( U^2 = S \), then \( b^2 + c^2 + d^2 = 1 \). Then, show that
\[
\exp^{-1}(R) = \begin{cases}
(2k + 1)\pi \begin{pmatrix}
0 & -d & c \\
d & 0 & -b \\
-c & b & 0
\end{pmatrix}, & k \in \mathbb{Z}
\end{cases},
\]
where \((b, c, d)\) is any unit vector such that for the corresponding skew symmetric matrix \( U \), we have \( U^2 = S \).

(e) To find a skew symmetric matrix \( U \) so that \( U^2 = S = \frac{1}{2} (R - I) \) as in (d), we can solve the system
\[
\begin{pmatrix}
b^2 - 1 & bc & bd \\
bc & c^2 - 1 & cd \\
bd & cd & d^2 - 1
\end{pmatrix} = S.
\]
We immediately get \( b^2, c^2, d^2 \), and then, since one of \( b, c, d \) is nonzero, say \( b \), if we choose the positive square root of \( b^2 \), we can determine \( c \) and \( d \) from \( bc \) and \( bd \).

Implement a computer program to solve the above system.

**Problem B5 (120 pts).** (a) Consider the set of affine maps \( \rho \) of \( \mathbb{R}^3 \) defined such that
\[
\rho(X) = \alpha RX + W,
\]
where \( R \) is a rotation matrix (an orthogonal matrix of determinant +1), \( W \) is some vector in \( \mathbb{R}^3 \), and \( \alpha \in \mathbb{R} \) with \( \alpha > 0 \). Every such a map can be represented by the \( 4 \times 4 \) matrix
\[
\begin{pmatrix}
\alpha R & W \\
0 & 1
\end{pmatrix}
\]
in the sense that
\[
\begin{pmatrix}
\rho(X) \\
1
\end{pmatrix} = \begin{pmatrix}
\alpha R & W \\
0 & 1
\end{pmatrix} \begin{pmatrix}
X \\
1
\end{pmatrix}
\]
iff
\[
\rho(X) = \alpha RX + W.
\]
Prove that these maps form a group, denoted by \( \text{SIM}(3) \) (the direct affine similitudes of \( \mathbb{R}^3 \)).
(b) Let us now consider the set of $4 \times 4$ real matrices of the form

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix},$$

where $\Gamma$ is a matrix of the form

$$\Gamma = \lambda I_3 + \Omega,$$

with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

so that

$$\Gamma = \begin{pmatrix} \lambda & -c & b \\ c & \lambda & -a \\ -b & a & \lambda \end{pmatrix},$$

and $W$ is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^7, +)$. This vector space is denoted by $\operatorname{sim}(3)$.

(c) Given a matrix

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Gamma^n & \Gamma^{n-1}W \\ 0 & 0 \end{pmatrix},$$

where $\Gamma^0 = I_3$. Prove that

$$e^B = \begin{pmatrix} e^\Gamma & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!}.$$

(d) Prove that if $\Gamma = \lambda I_3 + \Omega$ as in (b), then

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!} = \int_0^1 e^{\Gamma t} dt.$$

(e) For any matrix $\Gamma = \lambda I_3 + \Omega$, with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$
if we let $\theta = \sqrt{a^2 + b^2 + c^2}$, then prove that
\[ e^\Gamma = e^\lambda e^\Omega = e^\lambda \left( I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{(1 - \cos \theta)}{\theta^2} \Omega^2 \right), \quad \text{if } \theta \neq 0, \]
and $e^\Gamma = e^\lambda I_3$ if $\theta = 0$.

**Hint.** You may use the fact that if $AB = BA$, then $e^{A+B} = e^A e^B$. In general, $e^{A+B} \neq e^A e^B$!

(f) Prove that

1. If $\theta = 0$ and $\lambda = 0$, then $V = I_3$.
2. If $\theta = 0$ and $\lambda \neq 0$, then $V = \frac{(e^\lambda - 1)}{\lambda} I_3$;
3. If $\theta \neq 0$ and $\lambda = 0$, then $V = I_3 + \frac{(1 - \cos \theta)}{\theta^2} \Omega + \frac{(\theta - \sin \theta)}{\theta^3} \Omega^2$.
4. If $\theta \neq 0$ and $\lambda \neq 0$, then
\[ V = \frac{(e^\lambda - 1)}{\lambda} I_3 + \frac{\theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta}{\theta(\lambda^2 + \theta^2)} \Omega + \left( \frac{(e^\lambda - 1)}{\lambda \theta^2} - \frac{e^\lambda \sin \theta}{\theta(\lambda^2 + \theta^2)} - \frac{\lambda(e^\lambda \cos \theta - 1)}{\theta^2(\lambda^2 + \theta^2)} \right) \Omega^2. \]

**Hint.** You will need to compute $\int_0^1 e^{\lambda t} \sin \theta t \, dt$ and $\int_0^1 e^{\lambda t} \cos \theta t \, dt$.

(g) Prove that $V$ is invertible iff $\lambda \neq 0$ or $\theta \neq k2\pi$, with $k \in \mathbb{Z} - \{0\}$.

**Hint.** Express the eigenvalues of $V$ in terms of the eigenvalues of $\Gamma$.

In the special case where $\lambda = 0$, show that
\[ V^{-1} = I - \frac{1}{2} \Omega + \frac{1}{\theta^2} \left( 1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)} \right) \Omega^2, \quad \text{if } \theta \neq 0. \]

**Hint.** Assume that the inverse of $V$ is of the form
\[ Z = I_3 + a\Omega + b\Omega^2, \]
and show that $a, b$, are given by a system of linear equations that always has a unique solution.
(h) Prove that the exponential map \( \exp: \mathfrak{sim}(3) \to \text{SIM}(3) \), given by \( \exp(B) = e^B \), is surjective. You may use the fact that \( \exp: \mathfrak{so}(3) \to \text{SO}(3) \) is surjective, proved in another Problem.

Remark: As in the case of the plane, curves in \( \text{SIM}(3) \) can be used to describe certain deformations of bodies in \( \mathbb{R}^3 \).

Problem B6 (30 pts). Let \( E \) be a real vector space of finite dimension, \( n \geq 1 \). Say that two bases, \( (u_1, \ldots, u_n) \) and \( (v_1, \ldots, v_n) \), of \( E \) have the same orientation iff \( \det(P) > 0 \), where \( P \) the change of basis matrix from \( (u_1, \ldots, u_n) \) and \( (v_1, \ldots, v_n) \), namely, the matrix whose \( j \)th columns consist of the coordinates of \( v_j \) over the basis \( (u_1, \ldots, u_n) \).

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An orientation of a vector space, \( E \), is the choice of any fixed basis, say \( (e_1, \ldots, e_n) \), of \( E \). Any other basis, \( (v_1, \ldots, v_n) \), has the same orientation as \( (e_1, \ldots, e_n) \) (and is said to be positive or direct) iff \( \det(P) > 0 \), else it is said to have the opposite orientation of \( (e_1, \ldots, e_n) \) (or to be negative or indirect), where \( P \) is the change of basis matrix from \( (e_1, \ldots, e_n) \) to \( (v_1, \ldots, v_n) \). An oriented vector space is a vector space with some chosen orientation (a positive basis).

(b) Let \( B_1 = (u_1, \ldots, u_n) \) and \( B_2 = (v_1, \ldots, v_n) \) be two orthonormal bases. For any sequence of vectors, \( (w_1, \ldots, w_n) \), in \( E \), let \( \det_{B_1}(w_1, \ldots, w_n) \) be the determinant of the matrix whose columns are the coordinates of the \( w_j \)'s over the basis \( B_1 \) and similarly for \( \det_{B_2}(w_1, \ldots, w_n) \).

Prove that if \( B_1 \) and \( B_2 \) have the same orientation, then
\[
\det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n).
\]

Given any oriented vector space, \( E \), for any sequence of vectors, \( (w_1, \ldots, w_n) \), in \( E \), the common value, \( \det_B(w_1, \ldots, w_n) \), for all positive orthonormal bases, \( B \), of \( E \) is denoted
\[
\lambda_E(w_1, \ldots, w_n)
\]
and called a volume form of \( (w_1, \ldots, w_n) \).

(c) Given any Euclidean oriented vector space, \( E \), of dimension \( n \) for any \( n - 1 \) vectors, \( w_1, \ldots, w_{n-1} \), in \( E \), check that the map
\[
x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x)
\]
is a linear form. Then, prove that there is a unique vector, denoted \( w_1 \times \cdots \times w_{n-1} \), such that
\[
\lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x.
\]
for all \( x \in E \). The vector \( w_1 \times \cdots \times w_{n-1} \) is called the cross-product of \((w_1, \ldots, w_{n-1})\). It is a generalization of the cross-product in \( \mathbb{R}^3 \) (when \( n = 3 \)).

**Problem B7 (40 pts).** Given \( p \) vectors \((u_1, \ldots, u_p)\) in a Euclidean space \( E \) of dimension \( n \geq p \), the Gram determinant (or Gramian) of the vectors \((u_1, \ldots, u_p)\) is the determinant

\[
\text{Gram}(u_1, \ldots, u_p) = \begin{vmatrix}
\|u_1\|^2 & \langle u_1, u_2 \rangle & \ldots & \langle u_1, u_p \rangle \\
\langle u_2, u_1 \rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_p, u_1 \rangle & \langle u_p, u_2 \rangle & \ldots & \|u_p\|^2
\end{vmatrix}.
\]

1. Prove that 
\[
\text{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.
\]

*Hint.* If \((e_1, \ldots, e_n)\) is an orthonormal basis and \( A \) is the matrix of the vectors \((u_1, \ldots, u_n)\) over this basis,

\[
\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),
\]

where \( A^i \) denotes the \( i \)th column of the matrix \( A \), and \((A^i \cdot A^j)\) denotes the \( n \times n \) matrix with entries \( A^i \cdot A^j \).

2. Prove that 
\[
\|u_1 \times \cdots \times u_{n-1}\|^2 = \text{Gram}(u_1, \ldots, u_{n-1}).
\]

*Hint.* Letting \( w = u_1 \times \cdots \times u_{n-1} \), observe that

\[
\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,
\]

and show that

\[
\|w\|^4 = \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \text{Gram}(u_1, \ldots, u_{n-1}, w)
\]

\[
= \text{Gram}(u_1, \ldots, u_{n-1})\|w\|^2.
\]

**TOTAL:** 450 points.