Problem B1 (20 pts). Let $A$ be any real or complex $n \times n$ matrix and let $\| \|$ be any operator norm.

Prove that for every $m \geq 1$,

$$
\| I \| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e^{\| A \|}.
$$

Deduce from the above that the sequence $(E_m)$ of matrices

$$
E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}
$$

converges to a limit denoted $e^A$, and called the exponential of $A$.

Problem B2 (Extra Credit 60 pts). Recall that the affine maps $\rho: \mathbb{R}^2 \to \mathbb{R}^2$ defined such that

$$
\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},
$$

where $\theta, w_1, w_2 \in \mathbb{R}$, are rigid motions (or direct affine isometries) and that they form the group $\text{SE}(2)$.

Given any map $\rho$ as above, if we let

$$
R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},
$$

then $\rho$ can be represented by the $3 \times 3$ matrix

$$
A = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & w_1 \\ \sin \theta & \cos \theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}
$$
in the sense that
\[
\begin{pmatrix}
\rho(X)
\end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}
\]
iff
\[
\rho(X) = RX + W.
\]

(a) Consider the set of matrices of the form
\[
\begin{pmatrix}
0 & -\theta & u \\
\theta & 0 & v \\
0 & 0 & 0
\end{pmatrix}
\]
where \( \theta, u, v \in \mathbb{R} \). Verify that this set of matrices is a vector space isomorphic to \((\mathbb{R}^3, +)\). This vector space is denoted by \( \mathfrak{s}(2) \). Show that in general, \( BC \neq CB \), if \( B, C \in \mathfrak{s}(2) \).

(b) Given a matrix
\[
B = \begin{pmatrix} 0 & -\theta & u \\
\theta & 0 & v \\
0 & 0 & 0 \end{pmatrix},
\]
prove that if \( \theta = 0 \), then
\[
e^B = \begin{pmatrix} 1 & 0 & u \\
0 & 1 & v \\
0 & 0 & 1 \end{pmatrix},
\]
and that if \( \theta \neq 0 \), then
\[
e^B = \begin{pmatrix}
\cos \theta & -\sin \theta & \frac{u}{\theta} \sin \theta + \frac{v}{\theta} (\cos \theta - 1) \\
\sin \theta & \cos \theta & \frac{u}{\theta} (-\cos \theta + 1) + \frac{v}{\theta} \sin \theta \\
0 & 0 & 1
\end{pmatrix}.
\]

Hint. Prove that
\[
B^3 = -\theta^2 B,
\]
and that
\[
e^B = I_3 + \frac{\sin \theta}{\theta} B + \frac{1 - \cos \theta}{\theta^2} B^2.
\]

(c) Check that \( e^B \) is a direct affine isometry in \( \text{SE}(2) \). Prove that the exponential map \( \exp: \mathfrak{s}(2) \to \text{SE}(2) \) is surjective. If \( \theta \neq k2\pi \) \((k \in \mathbb{Z})\), how do you need to restrict \( \theta \) to get an injective map?

**Problem B3 (100 + 200 pts).** Consider the affine maps \( \rho: \mathbb{R}^2 \to \mathbb{R}^2 \) defined such that
\[
\rho \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},
\]
where $\theta, w_1, w_2, \alpha \in \mathbb{R}$, with $\alpha > 0$. These maps are called (direct) affine similitudes (for short, similitudes). The number $\alpha > 0$ is the scale factor of the similitude. These affine maps are the composition of a rotation of angle $\theta$, a rescaling by $\alpha > 0$, and a translation.

(a) Prove that these maps form a group that we denote by $\text{SIM}(2)$.

Given any map $\rho$ as above, if we let
\[
R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},
\]
then $\rho$ can be represented by the $3 \times 3$ matrix
\[
A = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha \cos \theta & -\alpha \sin \theta & w_1 \\ \alpha \sin \theta & \alpha \cos \theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}
\]
in the sense that
\[
\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}
\]
iff
\[
\rho(X) = \alpha RX + W.
\]

(b) Consider the set of matrices of the form
\[
\begin{pmatrix} \lambda & -\theta & u \\ \theta & \lambda & v \\ 0 & 0 & 0 \end{pmatrix}
\]
where $\theta, \lambda, u, v \in \mathbb{R}$. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^4, +)$. This vector space is denoted by $\text{sim}(2)$.

(c) Given a matrix
\[
\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},
\]
prove that
\[
e^\Omega = e^\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

Hint. Write
\[
\Omega = \lambda I + \theta J,
\]
with
\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Observe that $J^2 = -I$, and prove by induction on $k$ that
\[
\Omega^k = \frac{1}{2} \left( (\lambda + i\theta)^k + (\lambda - i\theta)^k \right) I + \frac{1}{2i} \left( (\lambda + i\theta)^k - (\lambda - i\theta)^k \right) J.
\]

(d) As in (c), write
\[
\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},
\]
let
\[
U = \begin{pmatrix} u \\ v \end{pmatrix},
\]
and let
\[
B = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}.
\]
Prove that
\[
B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}U \\ 0 & 0 \end{pmatrix}
\]
where $\Omega^0 = I_2$.

Prove that
\[
e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},
\]
where
\[
V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k + 1)!}.
\]

(e) Prove that
\[
V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k + 1)!} = \int_0^1 e^{\Omega t} dt.
\]
Use this formula to prove that if $\lambda = \theta = 0$, then
\[
V = I_2,
\]
else
\[
V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta & -\theta(1 - e^\lambda \cos \theta) - e^\lambda \lambda \sin \theta \\ \theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta & \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta \end{pmatrix}.
\]
Observe that for $\lambda = 0$, the above gives back the expression in B2(b) for $\theta \neq 0$.

Conclude that if $\lambda = \theta = 0$, then
\[
e^B = \begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix},
\]
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else

\[ e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix}, \]

with

\[ e^\Omega = e^\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \]

and

\[ V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta & -\theta(1 - e^\lambda \cos \theta) - e^\lambda \lambda \sin \theta \\ \theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta & \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta \end{pmatrix}, \]

and that \( e^B \in \text{SIM}(2) \), with scale factor \( e^\lambda \).

(f) Prove that the exponential map \( \exp: \text{sim}(2) \to \text{SIM}(2) \) is surjective.

(g) Similitudes can be used to describe certain deformations (or flows) of a deformable body \( B_t \) in the plane. Given some initial shape \( B \) in the plane (for example, a circle), a deformation of \( B \) is given by a piecewise differentiable curve

\[ D: [0, T] \to \text{SIM}(2), \]

where each \( D(t) \) is a similitude (for some \( T > 0 \)). The deformed body \( B_t \) at time \( t \) is given by

\[ B_t = D(t)(B). \]

The surjectivity of the exponential map \( \exp: \text{sim}(2) \to \text{SIM}(2) \) implies that there is a map \( \log: \text{SIM}(2) \to \text{sim}(2) \), although it is multivalued. The exponential map and the log “function” allows us to work in the simpler (noncurved) Euclidean space \( \text{sim}(2) \).

For instance, given two similitudes \( A_1, A_2 \in \text{SIM}(2) \) specifying the shape of \( B \) at two different times, we can compute \( \log(A_1) \) and \( \log(A_2) \), which are just elements of the Euclidean space \( \text{sim}(2) \), form the linear interpolant \((1 - t)\log(A_1) + t\log(A_2)\), and then apply the exponential map to get an interpolating deformation

\[ t \mapsto e^{(1-t)\log(A_1)+t\log(A_2)}, \quad t \in [0, 1]. \]

Also, given a sequence of “snapshots” of the deformable body \( B \), say \( A_0, A_1, \ldots, A_m \), where each is \( A_i \) is a similitude, we can try to find an interpolating deformation (a curve in \( \text{SIM}(2) \)) by finding a simpler curve \( t \mapsto C(t) \) in \( \text{sim}(2) \) (say, a B-spline) interpolating \( \log A_1, \log A_1, \ldots, \log A_m \). Then, the curve \( t \mapsto e^{C(t)} \) yields a deformation in \( \text{SIM}(2) \) interpolating \( A_0, A_1, \ldots, A_m \).

(1) (75 pts) Write a program interpolating between two deformations.

(2) (125 pts) Write a program using your cubic spline interpolation program from the first project, to interpolate a sequence of deformations given by similitudes \( A_0, A_1, \ldots, A_m \).
Problem B4 (40 pts). Recall that the coordinates of the cross product \( u \times v \) of two vectors \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) in \( \mathbb{R}^3 \) are
\[
(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).
\]

(a) If we let \( U \) be the matrix
\[
U = \begin{pmatrix}
0 & -u_3 & u_2 \\
u_3 & 0 & -u_1 \\
-u_2 & u_1 & 0
\end{pmatrix},
\]
check that the coordinates of the cross product \( u \times v \) are given by
\[
\begin{pmatrix}
0 & -u_3 & u_2 \\
u_3 & 0 & -u_1 \\
-u_2 & u_1 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= \begin{pmatrix}
0 & -u_3 & u_2 \\
u_3 & 0 & -u_1 \\
-u_2 & u_1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -u_3 & u_2 \\
u_3 & 0 & -u_1 \\
-u_2 & u_1 & 0
\end{pmatrix}
= \begin{pmatrix}
u_2v_3 - u_3v_2 \\
u_3v_1 - u_1v_3 \\
u_1v_2 - u_2v_1
\end{pmatrix}.
\]

(b) Show that the set of matrices of the form
\[
U = \begin{pmatrix}
0 & -u_3 & u_2 \\
u_3 & 0 & -u_1 \\
-u_2 & u_1 & 0
\end{pmatrix}
\]
is a vector space isomorphic to \( (\mathbb{R}^3^+) \). This vector space is denoted by \( \mathfrak{so}(3) \). Show that such matrices are never invertible. Find the kernel of the linear map associated with a matrix \( U \). Describe geometrically the action of the linear map defined by a matrix \( U \). Show that when restricted to the plane orthogonal to \( u = (u_1, u_2, u_3) \), if \( u \) is a unit vector, then \( U \) behaves like a rotation by \( \pi/2 \).

Problem B5 (100 pts). (a) For any matrix
\[
A = \begin{pmatrix}
0 & -c & b \\
c & 0 & -a \\
-b & a & 0
\end{pmatrix},
\]
if we let \( \theta = \sqrt{a^2 + b^2 + c^2} \) and
\[
B = \begin{pmatrix}
a^2 & ab & ac \\
ab & b^2 & bc \\
ac & bc & c^2
\end{pmatrix},
\]
prove that
\[
A^2 = -\theta^2 I + B,
AB = BA = 0.
\]
From the above, deduce that
\[ A^3 = -\theta^2 A. \]

(b) Prove that the exponential map \( \exp: \mathfrak{so}(3) \to \text{SO}(3) \) is given by
\[
\exp A = e^A = \cos \theta I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} B,
\]
or, equivalently, by
\[
e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2,
\]
if \( \theta \neq 0 \), with \( \exp(0_3) = I_3 \).

(c) Prove that \( e^A \) is an orthogonal matrix of determinant +1, i.e., a rotation matrix.

(d) Prove that the exponential map \( \exp: \mathfrak{so}(3) \to \text{SO}(3) \) is surjective. For this, proceed as follows: Pick any rotation matrix \( R \in \text{SO}(3) \);

(1) The case \( R = I \) is trivial.

(2) If \( R \neq I \) and \( \text{tr}(R) \neq -1 \), then
\[
\exp^{-1}(R) = \left\{ \frac{\theta}{2 \sin \theta} (R - R^T) \left| 1 + 2 \cos \theta = \text{tr}(R) \right. \right\}.
\]
(Recall that \( \text{tr}(R) = r_{11} + r_{22} + r_{33} \), the trace of the matrix \( R \)). Note that both \( \theta \) and \( 2\pi - \theta \) yield the same matrix \( \exp(R) \).

(3) If \( R \neq I \) and \( \text{tr}(R) = -1 \), then prove that the eigenvalues of \( R \) are 1, -1, -1, that \( R = R^\top \), and that \( R^2 = I \). Prove that the matrix
\[
S = \frac{1}{2}(R - I)
\]
is a symmetric matrix whose eigenvalues are -1, -1, 0. Thus, \( S \) can be diagonalized with respect to an orthogonal matrix \( Q \) as
\[
S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^\top.
\]
Prove that there exists a skew symmetric matrix
\[
U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}
\]
so that

\[ U^2 = S = \frac{1}{2}(R - I). \]

Observe that

\[ U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix}. \]

and use this to conclude that if \( U^2 = S \), then \( b^2 + c^2 + d^2 = 1 \). Then, show that

\[ \exp^{-1}(R) = \left\{ (2k + 1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}, \ k \in \mathbb{Z} \right\}, \]

where \((b, c, d)\) is any unit vector such that for the corresponding skew symmetric matrix \( U \), we have \( U^2 = S \).

(e) To find a skew symmetric matrix \( U \) so that \( U^2 = S = \frac{1}{2}(R - I) \) as in (d), we can solve the system

\[ \begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S. \]

We immediately get \( b^2, c^2, d^2 \), and then, since one of \( b, c, d \) is nonzero, say \( b \), if we choose the positive square root of \( b^2 \), we can determine \( c \) and \( d \) from \( bc \) and \( bd \).

Implement a computer program to solve the above system.

**Problem B6 (Extra Credit 100 pts).** (a) Consider the set of affine maps \( \rho \) of \( \mathbb{R}^3 \) defined such that

\[ \rho(X) = RX + W, \]

where \( R \) is a rotation matrix (an orthogonal matrix of determinant +1) and \( W \) is some vector in \( \mathbb{R}^3 \). Every such a map can be represented by the \( 4 \times 4 \) matrix

\[ \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \]

in the sense that

\[ \begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix} \]

iff

\[ \rho(X) = RX + W. \]

Prove that these maps are affine bijections and that they form a group, denoted by \( \text{SE}(3) \) (the *direct affine isometries, or rigid motions*, of \( \mathbb{R}^3 \)).
(b) Let us now consider the set of $4 \times 4$ matrices of the form

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix},$$

where $\Omega$ is a skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

and $W$ is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^6, +)$. This vector space is denoted by $\mathfrak{se}(3)$. Show that in general, $BC \neq CB$.

(c) Given a matrix

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix}$$
as in (b), prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}W \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_3$. Given

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

let $\theta = \sqrt{a^2 + b^2 + c^2}$. Prove that if $\theta = 0$, then

$$e^B = \begin{pmatrix} I_3 & W \\ 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$, then

$$e^B = \begin{pmatrix} e^{\Omega} & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$  

(d) Prove that

$$e^\Omega = I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{(1 - \cos \theta)}{\theta^2} \Omega^2$$
and

$$V = I_3 + \frac{(1 - \cos \theta)}{\theta^2} \Omega + \frac{(\theta - \sin \theta)}{\theta^3} \Omega^2.$$  

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Hint. Use the fact that $\Omega^3 = -\theta^2 \Omega$.

(e) Prove that $e^B$ is a direct affine isometry in $\mathbf{SE}(3)$. Prove that $V$ is invertible and that
\[ Z = I - \frac{1}{2} \Omega + \frac{1}{\theta^2} \left( 1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)} \right) \Omega^2, \]
for $\theta \neq 0$.

Hint. Assume that the inverse of $V$ is of the form
\[ Z = I_3 + a\Omega + b\Omega^2, \]
and show that $a, b$, are given by a system of linear equations that always has a unique solution.

Prove that the exponential map $\exp: \mathfrak{se}(3) \to \mathbf{SE}(3)$ is surjective.

Remark: As in the case of the plane, rigid motions in $\mathbf{SE}(3)$ can be used to describe certain deformations of bodies in $\mathbb{R}^3$.

Problem B7 (80 pts). Let $A$ be a real $2 \times 2$ matrix
\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \]

(1) Prove that the squares of the singular values $\sigma_1 \geq \sigma_2$ of $A$ are the roots of the quadratic equation
\[ X^2 - \text{tr}(A^\top A)X + |\det(A)|^2 = 0. \]

(2) If we let
\[ \mu(A) = \frac{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}{2|a_{11}a_{22} - a_{12}a_{21}|}, \]
prove that
\[ \text{cond}_{2}(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}. \]

(3) Consider the subset $\mathcal{S}$ of $2 \times 2$ invertible matrices whose entries $a_{ij}$ are integers such that $0 \leq a_{ij} \leq 100$.

Prove that the functions $\text{cond}_{2}(A)$ and $\mu(A)$ reach a maximum on the set $\mathcal{S}$ for the same values of $A$.

Check that for the matrix
\[ A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}, \]
we have
\[ \mu(A_m) = 19,603 \quad \det(A_m) = -1. \]
and
\[ \text{cond}_2(A_m) \approx 39,206. \]

(4) Prove that for all \( A \in \mathcal{S} \), if \( |\det(A)| \geq 2 \) then \( \mu(A) \leq 10,000 \). Conclude that the maximum of \( \mu(A) \) on \( \mathcal{S} \) is achieved for matrices such that \( \det(A) = \pm 1 \). Prove that finding matrices that maximize \( \mu \) on \( \mathcal{S} \) is equivalent to finding some integers \( n_1, n_2, n_3, n_4 \) such that
\begin{align*}
0 &\leq n_4 \leq n_3 \leq n_2 \leq n_1 \leq 100 \\
 n_1^2 + n_2^2 + n_3^2 + n_4^2 &\geq 100^2 + 99^2 + 99^2 + 98^2 = 39,206 \\
 |n_1n_4 - n_2n_3| &= 1.
\end{align*}

You may use without proof that the fact that the only solution to the above constraints is the multiset
\[ \{100, 99, 99, 98\}. \]

(5) Deduce from part (4) that the matrices in \( \mathcal{S} \) for which \( \mu \) has a maximum value are
\[
A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} 98 & 99 \\ 99 & 100 \end{pmatrix} \begin{pmatrix} 99 & 100 \\ 98 & 99 \end{pmatrix} \begin{pmatrix} 100 & 99 \end{pmatrix}
\]
and check that \( \mu \) has the same value for these matrices. Conclude that
\[ \max_{A \in \mathcal{S}} \text{cond}_2(A) = \text{cond}_2(A_m). \]

(6) Solve the system
\[
\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.
\]

Perturb the right-hand side \( b \) by
\[
\delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}
\]
and solve the new system
\[ A_m y = b + \delta b \]
where \( y = (y_1, y_2) \). Check that
\[
\delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.
\]
Compute \( \|x\|_2, \|\delta x\|_2, \|b\|_2, \|\delta b\|_2 \), and estimate
\[
c = \frac{\|\delta x\|_2}{\|x\|_2} \left( \frac{\|\delta b\|_2}{\|b\|_2} \right)^{-1}.
\]
Check that 
\[ c \approx \text{cond}_2(A_m) = 39,206. \]

**Problem B8 (20 pts).** Let \( E \) be a real vector space of finite dimension, \( n \geq 1 \). Say that two bases, \((u_1, \ldots, u_n)\) and \((v_1, \ldots, v_n)\), of \( E \) have the same orientation iff \( \det(P) > 0 \), where \( P \) the change of basis matrix from \((u_1, \ldots, u_n)\) and \((v_1, \ldots, v_n)\), namely, the matrix whose \( j \)th columns consist of the coordinates of \( v_j \) over the basis \((u_1, \ldots, u_n)\).

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An orientation of a vector space, \( E \), is the choice of any fixed basis, say \((e_1, \ldots, e_n)\), of \( E \). Any other basis, \((v_1, \ldots, v_n)\), has the same orientation as \((e_1, \ldots, e_n)\) (and is said to be positive or direct) iff \( \det(P) > 0 \), else it is said to have the opposite orientation of \((e_1, \ldots, e_n)\) (or to be negative or indirect), where \( P \) is the change of basis matrix from \((e_1, \ldots, e_n)\) to \((v_1, \ldots, v_n)\). An oriented vector space is a vector space with some chosen orientation (a positive basis).

(b) Let \( B_1 = (u_1, \ldots, u_n) \) and \( B_2 = (v_1, \ldots, v_n) \) be two orthonormal bases. For any sequence of vectors, \((w_1, \ldots, w_n)\), in \( E \), let \( \det_{B_1}(w_1, \ldots, w_n) \) be the determinant of the matrix whose columns are the coordinates of the \( w_j \)'s over the basis \( B_1 \) and similarly for \( \det_{B_2}(w_1, \ldots, w_n) \).

Prove that if \( B_1 \) and \( B_2 \) have the same orientation, then 
\[ \det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n). \]

Given any oriented vector space, \( E \), for any sequence of vectors, \((w_1, \ldots, w_n)\), in \( E \), the common value, \( \det_B(w_1, \ldots, w_n) \), for all positive orthonormal bases, \( B \), of \( E \) is denoted
\[ \lambda_E(w_1, \ldots, w_n) \]
and called a volume form of \((w_1, \ldots, w_n)\).

(c) Given any Euclidean oriented vector space, \( E \), of dimension \( n \) for any \( n - 1 \) vectors, \( w_1, \ldots, w_{n-1} \), in \( E \), check that the map
\[ x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x) \]
is a linear form. Then, prove that there is a unique vector, denoted \( w_1 \times \cdots \times w_{n-1} \), such that
\[ \lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x, \]
for all \( x \in E \). The vector \( w_1 \times \cdots \times w_{n-1} \) is called the cross-product of \((w_1, \ldots, w_{n-1})\). It is a generalization of the cross-product in \( \mathbb{R}^3 \) (when \( n = 3 \)).
Problem B9 (40 pts). Given $p$ vectors $(u_1, \ldots, u_p)$ in a Euclidean space $E$ of dimension $n \geq p$, the Gram determinant (or Gramian) of the vectors $(u_1, \ldots, u_p)$ is the determinant

$$\text{Gram}(u_1, \ldots, u_p) = \left| \begin{array}{cccc}
\|u_1\|^2 & \langle u_1, u_2 \rangle & \ldots & \langle u_1, u_p \rangle \\
\langle u_2, u_1 \rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle u_p, u_1 \rangle & \langle u_p, u_2 \rangle & \ldots & \|u_p\|^2 
\end{array} \right|.$$ 

(1) Prove that

$$\text{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.$$ 

*Hint.* If $(e_1, \ldots, e_n)$ is an orthonormal basis and $A$ is the matrix of the vectors $(u_1, \ldots, u_n)$ over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),$$

where $A^i$ denotes the $i$th column of the matrix $A$, and $(A^i \cdot A^j)$ denotes the $n \times n$ matrix with entries $A^i \cdot A^j$.

(2) Prove that

$$\|u_1 \times \cdots \times u_{n-1}\|^2 = \text{Gram}(u_1, \ldots, u_{n-1}).$$ 

*Hint.* Letting $w = u_1 \times \cdots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,$$

and show that

$$\|w\|^4 = \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \text{Gram}(u_1, \ldots, u_{n-1}, w)
= \text{Gram}(u_1, \ldots, u_{n-1})\|w\|^2.$$

Problem B10 (60 pts). (1) Implement the Gram-Schmidt orthonormalization procedure and the modified Gram-Schmidt procedure. You may use the pseudo-code showed in (2).

(2) Implement the method to compute the QR decomposition of an invertible matrix. You may use the following pseudo-code:

```plaintext
function qr(A: matrix): [Q, R] pair of matrices
begin
    n = dim(A);
    R = 0; (the zero matrix)
    Q1(:, 1) = A(:, 1);
    R(1, 1) = sqrt(Q1(:, 1)' * Q1(:, 1));
    Q(:, 1) = Q1(:, 1)/R(1, 1);
    for k := 1 to n - 1 do
```

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\[ w = A(:,k+1); \]
\[ \text{for } i := 1 \text{ to } k \text{ do} \]
\[ R(i,k+1) = A(:,k+1)^\top \cdot Q(:,i); \]
\[ w = w - R(i,k+1)Q(:,i); \]
\[ \text{endfor}; \]
\[ Q1(:,k+1) = w; \]
\[ R(k+1,k+1) = \sqrt{Q1(:,k+1)^\top \cdot Q1(:,k+1)}; \]
\[ Q(:,k+1) = Q1(:,k+1)/R(k+1,k+1); \]
\[ \text{endfor}; \]
end

Test it on various matrices, including those involved in Project 1.

(3) Given any invertible matrix \( A \), define the sequences \( A_k, Q_k, R_k \) as follows:

\[
\begin{align*}
A_1 &= A \\
Q_k R_k &= A_k \\
A_{k+1} &= R_k Q_k
\end{align*}
\]

for all \( k \geq 1 \), where in the second equation, \( Q_k R_k \) is the QR decomposition of \( A_k \) given by part (2).

Run the above procedure for various values of \( k \) (50, 100, ...) and various real matrices \( A \), in particular some symmetric matrices; also run the Matlab command \texttt{eig} on \( A_k \), and compare the diagonal entries of \( A_k \) with the eigenvalues given by \texttt{eig}(\( A_k \)).

What do you observe? How do you explain this behavior?

\textbf{TOTAL: 400 + 260 points + 160 extra.}