Problem B1 (40 pts). Let $A$ be an $n \times n$ matrix which is strictly row diagonally dominant, which means that
\[ |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \]
for $i = 1, \ldots, n$, and let
\[ \delta = \min_i \left\{ |a_{ii}| - \sum_{j \neq i} |a_{ij}| \right\}. \]
The fact that $A$ is strictly row diagonally dominant is equivalent to the condition $\delta > 0$.

(1) For any nonzero vector $v$, prove that
\[ \|Av\|_\infty \geq \|v\|_\infty \delta. \]
Use the above to prove that $A$ is invertible.

(2) Prove that
\[ \|A^{-1}\|_\infty \leq \delta^{-1}. \]

*Hint.* Prove that
\[ \sup_{v \neq 0} \frac{\|A^{-1}v\|_\infty}{\|v\|_\infty} = \sup_{w \neq 0} \frac{\|w\|_\infty}{\|Aw\|_\infty}. \]

Problem B2 (20 pts). Let $A$ be any invertible complex $n \times n$ matrix.

(1) For any vector norm $\| \|$ on $\mathbb{C}^n$, prove that the function $\| \|_A : \mathbb{C}^n \to \mathbb{R}$ given by
\[ \|x\|_A = \|Ax\| \quad \text{for all} \quad x \in \mathbb{C}^n, \]
is a vector norm.

(2) Prove that the operator norm induced by $\| \|_A$, also denoted by $\| \|_A$, is given by
\[ \|B\|_A = \|ABA^{-1}\| \quad \text{for every} \quad n \times n \text{ matrix} \quad B, \]
where $\|ABA^{-1}\|$ uses the operator norm induced by $\| \|$. 
Problem B3 (80 pts). (1) Implement the method for converting a rectangular matrix to reduced row echelon form.

(2) Use the above method to find the inverse of an invertible $n \times n$ matrix $A$, by applying it to the the $n \times 2n$ matrix $[AI]$ obtained by adding the $n$ columns of the identity matrix to $A$.

(3) Consider the matrix

$$A = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & n \\
2 & 3 & 4 & 5 & \cdots & n+1 \\
3 & 4 & 5 & 6 & \cdots & n+2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n & n+1 & n+2 & n+3 & \cdots & 2n-1
\end{pmatrix}.$$ 

Using your program, find the row reduced echelon form of $A$ for $n = 4, \ldots, 20$.

Also run the Matlab \texttt{rref} function and compare results.

Your program probably disagrees with \texttt{rref} even for small values of $n$. The problem is that some pivots are very small and the normalization step (to make the pivot 1) causes roundoff errors. Use a tolerance parameter to fix this problem.

What can you conjecture about the rank of $A$?

(4) Prove that the matrix $A$ has the following row reduced form:

$$R = \begin{pmatrix}
1 & 0 & -1 & -2 & \cdots & -(n-2) \\
0 & 1 & 2 & 3 & \cdots & n-1 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$ 

Deduce from the above that $A$ has rank 2.

\textit{Hint.} Some well chosen sequence of row operations.

(5) Use your program to show that if you add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of $A$ you get an invertible matrix! In fact, running the Matlab function \texttt{chol} should tell you that these matrices are SPD (symmetric, positive definite).

\textbf{Remark:} The above phenomenon will be explained in Problem B4. If you have a rigorous and simple explanation for this phenomenon, let me know!
**Problem B4 (120 pts).** The purpose of this problem is to prove that the characteristic polynomial of the matrix

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & n \\
2 & 3 & 4 & 5 & \cdots & n + 1 \\
3 & 4 & 5 & 6 & \cdots & n + 2 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
n & n + 1 & n + 2 & n + 3 & \cdots & 2n - 1
\end{pmatrix}
\]

is

\[
P_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - n^2\lambda - \frac{1}{12}n^2(n^2 - 1)\right).
\]

(1) Prove that the characteristic polynomial \(P_A(\lambda)\) is given by

\[
P_A(\lambda) = \lambda^{n-2}P(\lambda),
\]

with

\[
P(\lambda) = \begin{vmatrix}
\lambda - 1 & -2 & -3 & -4 & \cdots & -n + 3 & -n + 2 & -n + 1 & -n \\
-\lambda - 1 & \lambda - 1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1
\end{vmatrix}.
\]

(2) Prove that the sum of the roots \(\lambda_1, \lambda_2\) of the (degree two) polynomial \(P(\lambda)\) is

\[
\lambda_1 + \lambda_2 = n^2.
\]

The problem is thus to compute the product \(\lambda_1 \lambda_2\) of these roots. Prove that

\[
\lambda_1 \lambda_2 = P(0).
\]
(3) The problem is now to evaluate $d_n = P(0)$, where

$$
\begin{vmatrix}
-1 & -2 & -3 & -4 & \cdots & -n+3 & -n+2 & -n+1 & -n \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\
\end{vmatrix}
$$

$d_n = \begin{vmatrix}
1 & -2 & -3 & -4 & \cdots & -n+3 & -n+2 & -n+1 & -n \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\
\end{vmatrix}$

I suggest the following strategy: cancel out the first entry in row 1 and row 2 by adding a suitable multiple of row 3 to row 1 and row 2, and then subtract row 2 from row 1. Do this twice.

You will notice that the first two entries on row 1 and the first two entries on row 2 change, but the rest of the matrix looks the same, except that the dimension is reduced.

This suggests setting up a recurrence involving the entries $u_k, v_k, x_k, y_k$ in the determinant

$$
D_k = \begin{vmatrix}
u_k & x_k & -3 & -4 & \cdots & -n+k-3 & -n+k-2 & -n+k-1 & -n+k \\
v_k & y_k & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\
\end{vmatrix},
$$

starting with $k = 0$, with

$$u_0 = -1, \quad v_0 = -1, \quad x_0 = -2, \quad y_0 = -1,$$

and ending with $k = n - 2$, so that

$$d_n = D_{n-2} = \begin{vmatrix}u_{n-3} & x_{n-3} & -3 \\
v_{n-3} & y_{n-3} & -1 \\
1 & -2 & 1 \end{vmatrix} = \begin{vmatrix}u_{n-2} & x_{n-2} \\
v_{n-2} & y_{n-2} \end{vmatrix}.$$
Prove that we have the recurrence relations

\[
\begin{pmatrix}
    u_{k+1} \\
    v_{k+1} \\
    x_{k+1} \\
    y_{k+1}
\end{pmatrix}
= \begin{pmatrix}
    2 & -2 & 1 & -1 \\
    0 & 2 & 0 & 1 \\
    -1 & 1 & 0 & 0 \\
    0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    u_k \\
    v_k \\
    x_k \\
    y_k
\end{pmatrix}
+ \begin{pmatrix}
    0 \\
    0 \\
    -2 \\
    -1
\end{pmatrix}.
\]

These appear to be nasty affine recurrence relations, so we will use the trick to convert this affine map to a linear map.

(4) Consider the linear map given by

\[
\begin{pmatrix}
    u_{k+1} \\
    v_{k+1} \\
    x_{k+1} \\
    y_{k+1}
\end{pmatrix}
= \begin{pmatrix}
    2 & -2 & 1 & -1 \\
    0 & 2 & 0 & 1 \\
    -1 & 1 & 0 & 0 \\
    0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    u_k \\
    v_k \\
    x_k \\
    y_k
\end{pmatrix}
+ \begin{pmatrix}
    0 \\
    0 \\
    -2 \\
    -1
\end{pmatrix},
\]

and show that its action on \( u_k, v_k, x_k, y_k \) is the same as the affine action of part (3).

Use Matlab to find the eigenvalues of the matrix

\[
T = \begin{pmatrix}
    2 & -2 & 1 & -1 & 0 \\
    0 & 2 & 0 & 1 & 0 \\
    -1 & 1 & 0 & 0 & -2 \\
    0 & -1 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

You will be stunned!

Let \( N \) be the matrix given by

\[
N = T - I.
\]

Prove that \( N^4 = 0 \).

Use this to prove that

\[
T^k = I + kN + \frac{1}{2}k(k-1)N^2 + \frac{1}{6}k(k-1)(k-2)N^3,
\]

for all \( k \geq 0 \).

(5) Prove that

\[
\begin{pmatrix}
    u_k \\
    v_k \\
    x_k \\
    y_k
\end{pmatrix}
= T^k \begin{pmatrix}
    -1 \\
    -1 \\
    -2 \\
    -1
\end{pmatrix} = \begin{pmatrix}
    2 & -2 & 1 & -1 & 0 \\
    0 & 2 & 0 & 1 & 0 \\
    -1 & 1 & 0 & 0 & -2 \\
    0 & -1 & 0 & 0 & -1
\end{pmatrix}^k \begin{pmatrix}
    -1 \\
    -1 \\
    -2 \\
    -1
\end{pmatrix},
\]

5
for $k \geq 0$.

Prove that

$$T^k = \begin{pmatrix}
  k + 1 & -k(k + 1) & k & -k^2 & \frac{1}{6}(k - 1)k(2k - 7) \\
  0 & k + 1 & 0 & k & -\frac{1}{2}(k - 1)k \\
  -k & k^2 & 1 - k & (k - 1)k & -\frac{1}{2}k((k - 6)k + 11) \\
  0 & -k & 0 & 1 - k & \frac{1}{2}(k - 3)k \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

and thus,

$$\begin{pmatrix}
  u_k \\
  v_k \\
  x_k \\
  y_k
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{6}(2k^3 + 3k^2 - 5k - 6) \\
  -\frac{1}{2}(k^2 + 3k + 2) \\
  \frac{1}{3}(-k^3 + k - 6) \\
  \frac{1}{2}(k^2 + k - 2)
\end{pmatrix},$$

and that

$$\left| \begin{array}{cc}
  u_k & x_k \\
  v_k & y_k
\end{array} \right| = -1 - \frac{7}{3}k - \frac{23}{12}k^2 - \frac{2}{3}k^3 - \frac{1}{12}k^4.$$

As a consequence, prove that amazingly,

$$d_n = D_{n-2} = -\frac{1}{12}n^2(n^2 - 1).$$

(6) Prove that the characteristic polynomial of $A$ is indeed

$$P_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - n^2\lambda - \frac{1}{12}n^2(n^2 - 1)\right).$$

Use the above to show that the two nonzero eigenvalues of $A$ are

$$\lambda = \frac{n}{2} \left(n \pm \frac{\sqrt{3}}{3}\sqrt{4n^2 - 1}\right).$$

The negative eigenvalue $\lambda_1$ can also be expressed as

$$\lambda_1 = n^2\frac{(3 - 2\sqrt{3})}{6} \sqrt{1 - \frac{1}{4n^2}}.$$

Use this expression to explain the phenomenon in B3(5): If we add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of $A$ we get an invertible matrix. What about $0.077351n^2$? Try it!
Problem B5 (40 pts). A method for computing the $n$th root $x^{1/n}$ of a positive real number $x \in \mathbb{R}$, with $n \in \mathbb{N}$ a positive integer $n \geq 2$, proceeds as follows: Define the sequence $(x_k)$, where $x_0$ is any chosen positive real, and

$$x_{k+1} = \frac{1}{n} \left( (n-1)x_k + \frac{x}{x_k^{n-1}} \right), \quad k \geq 0.$$ 

(1) Implement the above method in Matlab, and test it for various input values of $x$, $x_0$, and of $n \geq 2$, by running successively your program for $m = 2, 3, \ldots, 100$ iterations. Have your program plot the points $(i, x_i)$ to watch how quickly the sequence converges.

Experiment with various choices of $x_0$. One of these choices should be $x_0 = x$. Compare your answers with the result of applying the of Matlab function $x \mapsto x^{1/n}$.

In some case, when $x_0$ is small, the number of iterations has to be at least 1000. Exhibit this behavior.

Problem B6 (80 pts). Refer to Problem B5 for the definition of the sequence $(x_k)$.

(1) Define the relative error $\epsilon_k$ as

$$\epsilon_k = \frac{x_k}{x^{1/n}} - 1, \quad k \geq 0.$$ 

Prove that

$$\epsilon_{k+1} = \frac{x^{(1-1/n)}}{nx_k^{n-1}} \left( \frac{(n-1)x_k^n}{x} - \frac{nx_k^{n-1}}{x^{(1-1/n)}} + 1 \right),$$ 

and then that

$$\epsilon_{k+1} = \frac{1}{n(\epsilon_k + 1)^{n-1}} \left( \epsilon_k (\epsilon_k + 1)^{n-2} ((n-1)\epsilon_k + (n-2)) + 1 - (\epsilon_k + 1)^{n-2} \right),$$ 

for all $k \geq 0$.

(2) Since

$$\epsilon_k + 1 = \frac{x_k}{x^{1/n}},$$ 

and since we assumed $x_0, x > 0$, we have $\epsilon_0 + 1 > 0$. We would like to prove that

$$\epsilon_k \geq 0, \quad \text{for all } k \geq 1.$$ 

For this, consider the variations of the function $f$ given by

$$f(u) = (n-1)u^n - nx^{1/n}u^{n-1} + x,$$

for $u \in \mathbb{R}$. 
Use the above to prove that \( f(u) \geq 0 \) for all \( u \geq 0 \). Conclude that
\[
\epsilon_k \geq 0, \quad \text{for all} \quad k \geq 1.
\]

(3) Prove that if \( n = 2 \), then
\[
0 \leq \epsilon_{k+1} = \frac{\epsilon_k^2}{2(\epsilon_k + 1)}, \quad \text{for all} \quad k \geq 0,
\]
else if \( n \geq 3 \), then
\[
0 \leq \epsilon_{k+1} \leq \frac{(n-1)}{n} \epsilon_k, \quad \text{for all} \quad k \geq 1.
\]
Prove that the sequence \( (x_k) \) converges to \( x^{1/n} \) for every initial value \( x_0 > 0 \).

(4) When \( n = 2 \), we saw in B6(3) that
\[
0 \leq \epsilon_{k+1} = \frac{\epsilon_k^2}{2(\epsilon_k + 1)}, \quad \text{for all} \quad k \geq 0.
\]
For \( n = 3 \), prove that
\[
\epsilon_{k+1} = \frac{2\epsilon_k^2(3/2 + \epsilon_k)}{3(\epsilon_k + 1)^2}, \quad \text{for all} \quad k \geq 0,
\]
and for \( n = 4 \), prove that
\[
\epsilon_{k+1} = \frac{3\epsilon_k^2}{4(\epsilon_k + 1)^3} \left(2 + \frac{8}{3} \epsilon_k + \epsilon_k^2\right), \quad \text{for all} \quad k \geq 0.
\]
Let \( \mu_3 \) and \( \mu_4 \) be the functions given by
\[
\mu_3(a) = \frac{3}{2} + a,
\]
\[
\mu_4(a) = 2 + \frac{8}{3} a + a^2,
\]
so that if \( n = 3 \), then
\[
\epsilon_{k+1} = \frac{2\epsilon_k^2 \mu_3(\epsilon_k)}{3(\epsilon_k + 1)^2}, \quad \text{for all} \quad k \geq 0,
\]
and if \( n = 4 \), then
\[
\epsilon_{k+1} = \frac{3\epsilon_k^2 \mu_4(\epsilon_k)}{4(\epsilon_k + 1)^3}, \quad \text{for all} \quad k \geq 0.
\]
Prove that
\[
a \mu_3(a) \leq (a + 1)^2 - 1, \quad \text{for all} \quad a \geq 0,
\]
and
\[
a \mu_4(a) \leq (a + 1)^3 - 1, \quad \text{for all} \quad a \geq 0.
\]
Let $\eta_{3,k} = \mu_3(\epsilon_1)\epsilon_k$ when $n = 3$, and $\eta_{4,k} = \mu_4(\epsilon_1)\epsilon_k$ when $n = 4$. Prove that

$$\eta_{3,k} + 1 \leq \frac{2}{3} \eta_{3,k}^2,$$

for all $k \geq 1$,

and

$$\eta_{4,k} + 1 \leq \frac{3}{4} \eta_{4,k}^2,$$

for all $k \geq 1$.

Deduce from the above that the rate of convergence of $\eta_{i,k}$ is very fast, for $i = 3, 4$ (and $k \geq 1$).

Remark: If we let $\mu_2(a) = a$ for all $a$ and $\eta_{2,k} = \epsilon_k$, then we proved that

$$\eta_{2,k} + 1 \leq \frac{1}{2} \eta_{2,k}^2,$$

for all $k \geq 1$.

Extra Credit (150 pt)

(5) Prove that for all $n \geq 2$, we have

$$\epsilon_{k+1} = \left(\frac{n - 1}{n}\right) \frac{\epsilon_k^2 \mu_n(\epsilon_k)}{(\epsilon_k + 1)^{n-1}},$$

for all $k \geq 0$,

where $\mu_n$ is given by

$$\mu_n(a) = \frac{1}{2} n + n^{-4} \sum_{j=1}^{n-4} \frac{1}{n-1} \left( (n-1) \left(\begin{array}{c} n-2 \\ j \end{array}\right) + (n-2) \left(\begin{array}{c} n-2 \\ j+1 \end{array}\right) - \left(\begin{array}{c} n-2 \\ j+2 \end{array}\right) \right) a^j$$

$$+ \frac{n(n-2)}{n-1} a^{n-3} + a^{n-2}.$$ 

Furthermore, prove that $\mu_n$ can be expressed as

$$\mu_n(a) = \frac{1}{2} n + n(n-2) a + \sum_{j=2}^{n-4} \frac{(j+1)n}{(j+2)(n-1)} \left(\begin{array}{c} n-1 \\ j \end{array}\right) a^j + \frac{n(n-2)}{n-1} a^{n-3} + a^{n-2}.$$ 

(6) Prove that for every $j$, with $1 \leq j \leq n - 1$, the coefficient of $a^j$ in $a\mu_n(a)$ is less than or equal to the coefficient of $a^j$ in $(a + 1)^{n-1} - 1$, and thus

$$a\mu_n(a) \leq (a + 1)^{n-1} - 1,$$

for all $a \geq 0$,

with strict inequality if $n \geq 3$. In fact, prove that if $n \geq 3$, then for every $j$, with $3 \leq j \leq n-2$, the coefficient of $a^j$ in $a\mu_n(a)$ is strictly less than the coefficient of $a^j$ in $(a + 1)^{n-1} - 1$, and if $n \geq 4$, this also holds for $j = 2$.

Let $\eta_{n,k} = \mu_n(\epsilon_1)\epsilon_k$ ($n \geq 2$). Prove that

$$\eta_{n,k}^2 \leq \left(\frac{n-1}{n}\right) \eta_{n,k}^2,$$

for all $k \geq 1$.

TOTAL: 380 + 150 points.