Problem B1 (20 pts). (1) Given two vectors in $\mathbb{R}^2$ of coordinates $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$ 

(2) Given three vectors in $\mathbb{R}^3$ of coordinates $(d_1 - a_1, d_2 - a_2, d_3 - a_3)$, $(c_1 - a_1, c_2 - a_2, c_3 - a_3)$, and $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$ 

Problem B2 (10 pts). If $A$ is an $n \times n$ symmetric matrix and $B$ is any $n \times n$ invertible matrix, prove that $A$ is positive definite iff $B^T AB$ is positive definite.

Problem B3 (80 pts). (1) Let $A$ be any invertible $2 \times 2$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Prove that there is an invertible matrix $S$ such that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & ad - bc \end{pmatrix},$$

where $S$ is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

Conclude that every matrix $A$ in $\text{SL}(2)$ (the group of invertible $2 \times 2$ matrices $A$ with $\det(A) = +1$) is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

For any $a \neq 0, 1$, give an explicit factorization as above for

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$
What is this decomposition for \( a = -1 \)?

(2) Recall that a rotation matrix \( R \) (a member of the group \( \text{SO}(2) \)) is a matrix of the form
\[
R = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]
Prove that if \( \theta \neq k\pi \) (with \( k \in \mathbb{Z} \)), any rotation matrix can be written as a product
\[
R = U L U,
\]
where \( U \) is upper triangular and \( L \) is lower triangular of the form
\[
U = \begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}, \quad L = \begin{pmatrix}
1 & 0 \\
v & 1
\end{pmatrix}.
\]

Therefore, every plane rotation (except a flip about the origin when \( \theta = \pi \)) can be written as the composition of three shear transformations!

(3) Recall that \( E_{i,d} \) is the diagonal matrix
\[
E_{i,d} = \text{diag}(1, \ldots, 1, d, 1, \ldots, 1),
\]
whose diagonal entries are all +1, except the \((i, i)\)th entry which is equal to \( d \).

Given any \( n \times n \) matrix \( A \), for any pair \((i, j)\) of distinct row indices \((1 \leq i, j \leq n)\), prove that there exist two elementary matrices \( E_1(i, j) \) and \( E_2(i, j) \) of the form \( E_{k,\ell,\beta} \), such that
\[
E_{j,-1} E_1(i, j) E_2(i, j) E_1(i, j) A = P(i, j) A,
\]
the matrix obtained from the matrix \( A \) by permuting row \( i \) and row \( j \). Equivalently, we have
\[
E_1(i, j) E_2(i, j) E_1(i, j) A = E_{j,-1} P(i, j) A,
\]
the matrix obtained from \( A \) by permuting row \( i \) and row \( j \) and multiplying row \( j \) by \(-1\).

Prove that for every \( i = 2, \ldots, n \), there exist four elementary matrices \( E_3(i, d), E_4(i, d), E_5(i, d), E_6(i, d) \) of the form \( E_{k,\ell,\beta} \), such that
\[
E_6(i) E_5(i) E_4(i) E_3(i) E_{n,d} = E_{i,d}.
\]

What happens when \( d = -1 \), that is, what kinds of simplifications occur?

Prove that all permutation matrices can be written as products of elementary operations of the form \( E_{k,\ell,\beta} \) and the operation \( E_{n,-1} \).

(4) Prove that for every invertible \( n \times n \) matrix \( A \), there is a matrix \( S \) such that
\[
SA = \begin{pmatrix}
I_{n-1} & 0 \\
0 & d
\end{pmatrix} = E_{n,d}.
\]

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with \( d = \det(A) \), and where \( S \) is a product of elementary matrices of the form \( E_{k,\ell;\beta} \).

In particular, every matrix in \( \text{SL}(n) \) (the group of invertible \( n \times n \) matrices \( A \) with \( \det(A) = +1 \)) can be written as a product of elementary matrices of the form \( E_{k,\ell;\beta} \). Prove that at most \( n(n+1) - 2 \) such transformations are needed.

**Problem B4 (40 pts).** A matrix, \( A \), is called strictly column diagonally dominant iff

\[
|a_{jj}| > \sum_{i=1, i \neq j}^{n} |a_{ij}|, \quad \text{for } j = 1, \ldots, n
\]

Prove that if \( A \) is strictly column diagonally dominant, then Gaussian elimination does not require pivoting and \( A \) is invertible.

**Problem B5 (40 pts).** Let \((\alpha_1, \ldots, \alpha_{m+1})\) be a sequence of pairwise distinct scalars in \( \mathbb{R} \) and let \((\beta_1, \ldots, \beta_{m+1})\) be any sequence of scalars in \( \mathbb{R} \), not necessarily distinct.

(1) Prove that there is a unique polynomial \( P \) of degree at most \( m \) such that

\[
P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m + 1.
\]

*Hint.* Remember Vandermonde!

(2) Let \( L_i(X) \) be the polynomial of degree \( m \) given by

\[
L_i(X) = \frac{(X - \alpha_1) \cdots (X - \alpha_i - 1)(X - \alpha_i + 1) \cdots (X - \alpha_{m+1})}{(\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_i - 1)(\alpha_i - \alpha_i + 1) \cdots (\alpha_i - \alpha_{m+1})}, \quad 1 \leq i \leq m + 1.
\]

The polynomials \( L_i(X) \) are known as Lagrange polynomial interpolants. Prove that

\[
L_i(\alpha_j) = \delta_{ij}, \quad 1 \leq i, j \leq m + 1.
\]

Prove that

\[
P(X) = \beta_1 L_1(X) + \cdots + \beta_{m+1} L_{m+1}(X)
\]

is the unique polynomial of degree at most \( m \) such that

\[
P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m + 1.
\]

(3) Prove that \( L_1(X), \ldots, L_{m+1}(X) \) are linearly independent, and that they form a basis of all polynomials of degree at most \( m \).

How is 1 (the constant polynomial 1) expressed over the basis \((L_1(X), \ldots, L_{m+1}(X))\)?

Give the expression of every polynomial \( P(X) \) of degree at most \( m \) over the basis \((L_1(X), \ldots, L_{m+1}(X))\).

(4) Prove that the dual basis \((L_1^*, \ldots, L_{m+1}^*)\) of the basis \((L_1(X), \ldots, L_{m+1}(X))\) consists of the linear forms \( L_i^* \) given by

\[
L_i^*(P) = P(\alpha_i),
\]
for every polynomial \( P \) of degree at most \( m \); this is simply \( \text{evaluation at } \alpha_i \).

**Problem B6 (30 pts).** Consider the \( n \times n \) symmetric matrix

\[
A = \begin{pmatrix}
1 & 2 & 0 & 0 & \ldots & 0 & 0 \\
2 & 5 & 2 & 0 & \ldots & 0 & 0 \\
0 & 2 & 5 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 2 & 5 & 2 & 0 \\
0 & 0 & \ldots & 0 & 2 & 5 & 2 \\
0 & 0 & \ldots & 0 & 0 & 2 & 5
\end{pmatrix}
\]

(1) Find an upper-triangular matrix \( R \) such that \( A = R^T R \).

(2) Prove that \( \det(A) = 1 \).

(3) Consider the sequence

\[
p_0(\lambda) = 1 \\
p_1(\lambda) = 1 - \lambda \\
p_k(\lambda) = (5 - \lambda)p_{k-1}(\lambda) - 4p_{k-2}(\lambda) \quad 2 \leq k \leq n.
\]

Prove that

\[ \det(A - \lambda I) = p_n(\lambda). \]

**Remark:** It can be shown that \( p_n(\lambda) \) has \( n \) distinct (real) roots and that the roots of \( p_k(\lambda) \) separate the roots of \( p_{k+1}(\lambda) \).

**TOTAL: 220 points.**