Problem B1 (10 pts). Given any $m \times n$ matrix $A$ and any $n \times p$ matrix $B$, if we denote the columns of $A$ by $A^1, \ldots, A^n$ and the rows of $B$ by $B_1, \ldots, B_n$, prove that

$$AB = A^1 B_1 + \cdots + A^n B_n.$$

Problem B2 (10 pts). Let $f: E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1}: F \to E$ is linear.

Problem B3 (10 pts). Given two vectors spaces $E$ and $F$, let $(u_i)_{i \in I}$ be any basis of $E$ and let $(v_i)_{i \in I}$ be any family of vectors in $F$. Prove that the unique linear map $f: E \to F$ such that $f(u_i) = v_i$ for all $i \in I$ is surjective iff $(v_i)_{i \in I}$ spans $F$.

Problem B4 (10 pts). Let $f: E \to F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that $f$ has rank 1 iff $f$ is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with $u$ a nonzero column vector of dimension $m$ and $v$ a nonzero column vector of dimension $n$.

Problem B5 (120 pts). (Haar extravaganza) Consider the matrix

$$W_{3,3} = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
\end{pmatrix}$$

(1) Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,3}c$ of applying $W_{3,3}$ to $c$ is

$$W_{3,3}c = (c_1 + c_5, c_1 - c_5, c_2 + c_6, c_2 - c_6, c_3 + c_7, c_3 - c_7, c_4 + c_8, c_4 - c_8).$$
the last step in reconstructing a vector from its Haar coefficients.

(2) Prove that the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^T$. Prove that the columns and the rows of $W_{3,3}$ are orthogonal.

(3) Let $W_{3,2}$ and $W_{3,1}$ be the following matrices:

\[
W_{3,2} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

\[
W_{3,1} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,2}c$ of applying $W_{3,2}$ to $c$ is

\[
W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8),
\]

the second step in reconstructing a vector from its Haar coefficients, and the result $W_{3,1}c$ of applying $W_{3,1}$ to $c$ is

\[
W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8),
\]

the first step in reconstructing a vector from its Haar coefficients.

Conclude that

\[
W_{3,3}W_{3,2}W_{3,1} = W_3,
\]

the Haar matrix

\[
W_3 = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

Hint. First, check that

\[
W_{3,2}W_{3,1} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},
\]

where

\[
W_2 = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & -1 & 0 \\
1 & -1 & 0 & 1 \\
1 & -1 & 0 & -1 \\
\end{pmatrix}.
\]
(4) Prove that the columns and the rows of $W_{3,2}$ and $W_{3,1}$ are orthogonal. Deduce from this that the columns of $W_3$ are orthogonal, and the rows of $W_3^{-1}$ are orthogonal. Are the rows of $W_3$ orthogonal? Are the columns of $W_3^{-1}$ orthogonal? Find the inverse of $W_{3,2}$ and the inverse of $W_{3,1}$.

(5) For any $n \geq 2$, the $2^n \times 2^n$ matrix $W_{n,n}$ is obtained form the two rows

\[
\begin{align*}
1,0,\ldots,0, & 1,0,\ldots,0 \\
2^{n-1}, & 2^{n-1} \\
1,0,\ldots,0, & 0,1,0,\ldots,0 \\
2^{n-1}, & 2^{n-1}
\end{align*}
\]

by shifting them $2^{n-1} - 1$ times over to the right by inserting a zero on the left each time.

Given any vector $c = (c_1, c_2, \ldots, c_{2^n})$, show that $W_{n,n}c$ is the result of the last step in the process of reconstructing a vector from its Haar coefficients $c$. Prove that $W_{n,n}^{-1} = (1/2)W_{n,n}^\top$, and that the columns and the rows of $W_{n,n}$ are orthogonal.

**Extra credit (30 pts.)**

Given a $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix $B = (b_{ij})$, the Kronecker product (or tensor product) $A \otimes B$ of $A$ and $B$ is the $mp \times nq$ matrix

\[
A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix}.
\]

It can be shown (and you may use these facts without proof) that $\otimes$ is associative and that

\[
(A \otimes B)(C \otimes D) = AC \otimes BD
\]
\[
(A \otimes B)^\top = A^\top \otimes B^\top,
\]

whenever $AC$ and $BD$ are well defined.

Check that

\[
W_{n,n} = \left( I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \left( I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),
\]

and that

\[
W_n = \left( W_{n-1} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \left( I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).
\]

Use the above to reprove that

\[
W_{n,n}W_{n,n}^\top = 2I_{2^n}.
\]
Let 
\[ B_1 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \]
and for \( n \geq 1 \),
\[ B_{n+1} = 2 \begin{pmatrix} B_n & 0 \\ 0 & I_{2^n} \end{pmatrix} . \]

Prove that 
\[ W_n^T W_n = B_n, \quad \text{for all } n \geq 1. \]

(6) The matrix \( W_{n,i} \) is obtained from the matrix \( W_{i,i} \) \( (1 \leq i \leq n-1) \) as follows:
\[ W_{n,i} = \begin{pmatrix} W_{i,i} & 0_{2^i,2^{n-2i}} \\ 0_{2^{n-2i},2^i} & I_{2^n-2^i} \end{pmatrix} . \]

It consists of four blocks, where \( 0_{2^i,2^{n-2i}} \) and \( 0_{2^{n-2i},2^i} \) are matrices of zeros and \( I_{2^n-2^i} \) is the identity matrix of dimension \( 2^n - 2^i \).

Explain what \( W_{n,i} \) does to \( c \) and prove that
\[ W_{n,n} W_{n,n-1} \cdots W_{n,1} = W_n, \]
where \( W_n \) is the Haar matrix of dimension \( 2^n \).

Hint. Use induction on \( k \), with the induction hypothesis
\[ W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k,2^{n-2^k}} \\ 0_{2^{n-2^k},2^k} & I_{2^n-2^k} \end{pmatrix} . \]

Problem B6 (20 pts). Prove that for every vector space \( E \), if \( f : E \to E \) is an idempotent linear map, i.e., \( f \circ f = f \), then we have a direct sum
\[ E = \text{Ker } f \oplus \text{Im } f, \]
so that \( f \) is the projection onto its image \( \text{Im } f \).

Problem B7 (20 pts). Let \( U_1, \ldots, U_p \) be any \( p \geq 2 \) subspaces of some vector space \( E \) and recall that the linear map
\[ a : U_1 \times \cdots \times U_p \to E \]
is given by
\[ a(u_1, \ldots, u_p) = u_1 + \cdots + u_p, \]
with \( u_i \in U_i \) for \( i = 1, \ldots, p \).

(1) If we let \( Z_i \subseteq U_1 \times \cdots \times U_p \) be given by
\[
Z_i = \left\{ \left( u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_p \right) \left| \sum_{j=1, j \neq i}^{p} u_j, u_i \in U_i \cap \left( \sum_{j=1, j \neq i}^{p} U_j \right) \right. \right\},
\]
for \( i = 1, \ldots, p \), then prove that
\[
\text{Ker} \ a = Z_1 = \cdots = Z_p.
\]

In general, for any given \( i \), the condition \( U_i \cap \left( \sum_{j=1, j \neq i}^{p} U_j \right) = (0) \) does not necessarily imply that \( Z_i = (0) \). Thus, let
\[
Z = \left\{ \left( u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_p \right) \left| u_i = - \sum_{j=1, j \neq i}^{p} u_j, u_i \in U_i \cap \left( \sum_{j=1, j \neq i}^{p} U_j \right) \right. \right\},
\]
Since \( \text{Ker} \ a = Z_1 = \cdots = Z_p \), we have \( Z = \text{Ker} \ a \). Prove that if
\[
U_i \cap \left( \sum_{j=1, j \neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p,
\]
then \( Z = \text{Ker} \ a = (0) \).

(2) Prove that \( U_1 + \cdots + U_p \) is a direct sum iff
\[
U_i \cap \left( \sum_{j=1, j \neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p.
\]

(3) Extra credit (40 pts). Assume that \( E \) is finite-dimensional, and let \( f_i : E \to E \) be any \( p \geq 2 \) linear maps such that
\[
f_1 + \cdots + f_p = \text{id}_E.
\]
Prove that the following properties are equivalent:

(1) \( f_i^2 = f_i \), \( 1 \leq i \leq p \).

(2) \( f_j \circ f_i = 0 \), for all \( i \neq j, 1 \leq i, j \leq p \).

**TOTAL: 200 + 70 points.**