Problem B1 (10 pts). Given any $m \times n$ matrix $A$ and any $n \times p$ matrix $B$, if we denote the columns of $A$ by $A_1, \ldots, A_n$ and the rows of $B$ by $(B_1^\top)^\top, \ldots, (B_n^\top)^\top$, prove that

$$AB = A_1(B_1^\top)^\top + \cdots + A_n(B_n^\top)^\top.$$  

Problem B2 (10 pts). Let $f: E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1}: F \to E$ is linear.

Problem B3 (40 pts). Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space $E$ and recall that the linear map

$$a: U_1 \times \cdots \times U_p \to E$$

is given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$.

1. If we let $Z_i \subseteq U_1 \times \cdots \times U_p$ be given by

$$Z_i = \left\{(u_1, \ldots, u_{i-1}, -\sum_{j=1, j\neq i}^p u_j, u_{i+1}, \ldots, u_p) \mid u_j \in U_j, j \in \{1, \ldots, p\} - \{i\}\right\}$$

for $i = 1, \ldots, p$, then prove that

$$\text{Ker } a = Z_1 + \cdots + Z_p.$$  

2. Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1, j\neq i}^p U_j\right) = (0) \quad 1 \leq i \leq p.$$  

3. Let $f_i: E \to E$ be any $p \geq 2$ linear maps such that

$$f_1 + \cdots + f_p = \text{id}_E.$$  

Prove that the following properties are equivalent:
(1) $f_i^2 = f_i$, $1 \leq i \leq p$.

(2) $f_j \circ f_i = 0$, for all $i \neq j$, $1 \leq i, j \leq p$.

**Problem B4 (20 pts).** Prove that for every vector space $E$, if $f : E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \text{Ker } f \oplus \text{Im } f,$$

so that $f$ is the projection onto its image $\text{Im } f$.

**Problem B5 (10 pts).** Given any $n \times n$ matrix $A$, prove that multiplying $A$ by the elementary matrix $E_{i,j;\beta}$ on the right yields the matrix $AE_{i,j;\beta}$ in which $\beta$ times the $i$th column is added to the $j$th column.

**Problem B6 (50 pts).**

(a) Find an upper triangular matrix $E$ such that

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(b) What is the effect of the product (on the left) with

$$E_{4,3;1}E_{3,2;1}E_{4,3;1}E_{2,1;1}E_{3,2;1}E_{4,3;1}$$

on the matrix

$$P_{a3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(c) Find the inverse of the matrix $P_{a3}$.

(d) Consider the $(n + 1) \times (n + 1)$ Pascal matrix $P_{a_n}$ whose $i$th row is given by the binomial coefficients

$$\begin{pmatrix} i - 1 \\ j - 1 \end{pmatrix},$$

with $1 \leq i \leq n + 1, 1 \leq j \leq n + 1$, and with the usual convention that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1, \quad \begin{pmatrix} i \\ j \end{pmatrix} = 0 \quad \text{if} \quad j > i.$$

The matrix $P_{a3}$ is shown in question (c) and $P_{a4}$ is shown below:

$$P_{a4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$
Find $n$ elementary matrices $E_{i_k,j_k;\beta_k}$ such that

$$E_{i_n,j_n;\beta_n} \cdots E_{i_1,j_1;\beta_1} P_{a_n} = \begin{pmatrix} 1 & 0 \\ 0 & P_{a_{n-1}} \end{pmatrix}.$$ 

Use the above to prove that the inverse of $P_{a_n}$ is the lower triangular matrix whose $i$th row is given by the signed binomial coefficients

$$(-1)^{i+j-2} \binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$. For example,

$$P_{a_4}^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 \\
-1 & 3 & -3 & 1 & 0 \\
1 & -4 & 6 & -4 & 1
\end{pmatrix}.$$

**Problem B7 (10 pts).** Let $f : E \to F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that $f$ has rank 1 iff $f$ is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with $u$ a nonzero column vector of dimension $m$ and $v$ a nonzero column vector of dimension $n$.

**TOTAL: 150 points.**