“A problems” are for practice only, and should not be turned in.

**Problem A1.** Given an alphabet \( \Sigma \), prove that the relation \( \leq_1 \) over \( \Sigma^* \) defined such that 
\[ u \leq_1 v \text{ iff } u \text{ is a prefix of } v, \]
is a partial ordering. Prove that the relation \( \leq_2 \) over \( \Sigma^* \) defined
such that 
\[ u \leq_2 v \text{ iff } u \text{ is a substring of } v, \]
is a partial ordering.

**Problem A2.** Given an alphabet \( \Sigma \), for any language \( L \subseteq \Sigma^* \), prove that 
\[ L^{**} = L^* \]
and 
\[ L^* L^* = L^*. \]

**Problem A3.** Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a DFA. Prove that for all \( p \in Q \) and all \( u, v \in \Sigma^* \),
\[ \delta^*(p, uv) = \delta^*(\delta^*(p, u), v). \]

“B problems” must be turned in.

**Problem B1 (30 pts).** Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a DFA. Recall that a state \( p \in Q \) is
accessible or reachable iff there is some string \( w \in \Sigma^* \), such that
\[ \delta^*(q_0, w) = p, \]
i.e., there is some path from \( q_0 \) to \( p \) in \( D \). Consider the following method for computing the
set \( Q_r \) of reachable states (of \( D \)): define the sequence of sets \( Q^i_r \subseteq Q \), where
\[ Q^0_r = \{q_0\}, \]
\[ Q^{i+1}_r = \{q \in Q \mid \exists p \in Q^i_r, \exists a \in \Sigma, q = \delta(p, a)\}. \]
(i) Prove by induction on \( i \) that \( Q^i_r \) is the set of all states reachable from \( q_0 \) using paths
of length \( i \) (where \( i \) counts the number of edges).

Give an example of a DFA such that \( Q^{i+1}_r \neq Q^i_r \) for all \( i \geq 0 \).
(ii) Give an example of a DFA such that \( Q^i_r \neq Q_r \) for all \( i \geq 0 \).
(iii) Change the inductive definition of \( Q^i_r \) as follows:
\[ Q^{i+1}_r = Q^i_r \cup \{q \in Q \mid \exists p \in Q^i_r, \exists a \in \Sigma, q = \delta(p, a)\}. \]
Prove that there is a smallest integer \( i_0 \) such that
\[
Q_r^{i_0+1} = Q_r^{i_0} = Q_r.
\]

Define the DFA \( D_r \) as follows: \( D_r = (Q_r, \Sigma, \delta_r, q_0, F \cap Q_r) \), where \( \delta_r : Q_r \times \Sigma \to Q_r \) is the restriction of \( \delta \) to \( Q_r \). Explain why \( D_r \) is indeed a DFA, and prove that \( L(D_r) = L(D) \). A DFA is said to be reachable, or trim, if \( D = D_r \).

**Problem B2 (20 pts).** Given a string \( w \), its reversal \( w^R \) is defined inductively as follows:
\[
\epsilon^R = \epsilon \quad \text{and} \quad (ua)^R = au^R, \quad \text{where} \quad a \in \Sigma \quad \text{and} \quad u \in \Sigma^*.\]
Prove that \( (uv)^R = v^Ru^R \).

**Problem B3 (20 pts).** Construct DFA’s for the following languages:
(a) \( \{ w \mid w \in \{a, b\}^*, \ w \text{ has neither } aa \text{ nor } bb \text{ as a substring} \} \).
(b) \( \{ w \mid w \in \{a, b\}^*, \ w \text{ has an even number of } a \text{’s and an odd number of } b \text{’s} \} \).

**Problem B4 (30 pts).** Let \( L \) be a regular language. Are the following languages regular, and if so, give a proof (or construction).
(a) \( \text{Pre}(L) = \{ u \mid u \text{ is a prefix of some } w \in L \} \)
(b) \( \text{Suf}(L) = \{ u \mid u \text{ is a suffix of some } w \in L \} \)
(c) \( \text{Sub}(L) = \{ u \mid u \text{ is a substring of some } w \in L \} \)

**Problem B5 (20 pts).** Let \( L \) be any language over some alphabet \( \Sigma \).
(a) Prove that \( L = L^+ \) iff \( LL \subseteq L \).
(b) Prove that \( (L = \emptyset \text{ or } L = L^*) \) iff \( LL = L \).

**Problem B6 (40 pts).** Given any two relatively prime integers \( p, q \geq 1 \), with \( p \neq q \), \( p \) and \( q \) are relatively prime iff their greatest common divisor is 1), consider the language \( L = \{a^p, a^q\}^* \). Prove that
\[
\{a^p, a^q\}^* = \{a^n \mid n \geq (p-1)(q-1) \} \cup F,
\]
where \( F \) is some finite set of strings (of length < \( (p-1)(q-1) \)). Prove that \( L \) is a regular language.

**Extra Credit (20 pts).** Given any two relatively prime integers \( p, q \geq 1 \), with \( p \neq q \), prove that \( pq - p - q = (p-1)(q-1) - 1 \) is the largest integer not expressible as \( ph + kq \) with \( h, k \geq 0 \).

**Problem B7 (20 pts).** (a) Given the alphabet \( \Sigma = \{0, 1, c\} \), construct a DFA accepting the following language:
\[
L = \{ u_1cu_2c \cdots cu_{n-1}cu_n \mid n \geq 1, \ u_i \in \{00, 01, 10\} \}.
\]
(b) The strings in the above language can be interpreted as the coordinates of points in the plane as follows: Assume that you start with a square \( S_0 \), say of dimension \( 2 \times 2 \), divided
into four equal subsquares. Then the lower left corner of each subsquare is referenced by one of the strings 00, 01, 10, or 11. A string $u_1u_2\ldots u_{n-1}u_n$ determines a point in the original square by proceeding recursively as follows: $u_1$ determines the subsquare $S_1$ whose lower left corner has coordinates $u_1$ in the original square; within the square $S_1$, $u_2$ determines the subsquare $S_2$ whose lower left corner has coordinates $u_2$; given the subsquare $S_i$ obtained at the end of step $i$, within this subsquare $S_i$, $u_{i+1}$ determines the subsquare $S_{i+1}$ whose lower left corner has coordinates $u_{i+1}$. The procedure stops with a point in the square $S_{n-1}$ obtained at stage $n-1$, the lower left corner of the subsquare $S_n$ whose coordinates with respect to $S_{n-1}$ are determined by $u_n$.

Draw a rough picture by plotting a number of these points. What sort of shape do you get?

Remark: The set of points defined above is a subset of the set of rational points of a fractal set known as the Sierpinski gasket.

Extra Credit (30 pts). Write a computer program to display the Sierpinski gasket.

TOTAL: 180 + 50 points.