Inducing Approximately Optimal Flow Using Truthful Mediators

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Joint work with Ryan Rogers, Aaron Roth, and Jonathan Ullman
Traffic is Everywhere

- Road Network
- Internet Network
- One of the reasons: Selfish Routing
Atomic Routing Game

- A graph $G = (V, E)$, $|E| = m$
- Convex latency function for each edge $\ell_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+$
- $n$ players with their source-destination pairs $t_i = (s_i, d_i) \in V \times V$
- Each player routes a 1-unit of flow from his source to destination $f^i = (f_e^i)_{e \in E} \in \{0, 1\}^m$
- Aggregate flow $f = \sum_{i=1}^{n} f^i$
- Cost on each edge $\ell_e(f_e)$
- Cost for each player $c_i(f) = \sum_{e \in E} f_e^i \cdot \ell_e(f_e)$
Equilibrium Flow

- Approximate Equilibrium Flow $f$ satisfies:
  For any $i$ and $(s_i, d_i)$ flow $f'$, $c_i(f^i, f^{-i}) \leq c_i(f', f^{-i}) + \eta$

- Equilibrium flow minimizes the potential function [MS’96]
  $$\Psi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} \ell_e(i)$$

- Social cost objective
  $$C(f) = \sum_{e \in E} f_e \cdot \ell_e(f_e)$$
Inefficiency of Selfish Routing

- The price of anarchy is unboundedly large when the latencies can be arbitrary convex function
  [RoughgardenTardos’02]
Tolling to Rescue

Marginal-Cost Toll [BMW’56]

\[ \tau_e^*(f_e) = (f_e - 1) (\ell_e(f_e) - \ell_e(f_e - 1)) \]

New Latency Function

\[ \ell_e^*(f) = \ell_e(f) + \tau_e^*(f) \]

Lemma: Potential function of tolled game = Social cost function

\[ \Psi^*(f) = \sum_{e \in E} \sum_{i=1}^{f_e} \ell_e^*(i) = C(f) \]

Full Efficiency!
Two Problems

❖ The agents’ source/destination pairs are unknown. The mechanism needs to elicit the demands of the agents.

❖ Marginal-cost tolls are *functional* tolls: difficult to charge agents tolls as a function of what others are doing. (Ideally, use *fixed* tolls)
Mediator

Submit Source-Destination

Post Tolls

Recommend Paths
Mediated Game

❖ Each player has private type \((s, d)\)

❖ Action set:
  ❖ *opt-out* from using the mediator, and take some \((s, d)\)-path
  ❖ *opt-in* to using the mediator, report some source-destination pair (not necessarily \((s, d)\))
  ❖ upon receiving the suggestion, follow the suggestion or deviate based on the suggestion
Good Behavior

- Good behavior:
  - *truthfully* report the source-destination and
  - *faithfully* follow the suggested action of the mediator

- Goal for the mediator:
  - incentives *good behavior* of the players
  - suggests approximately *min-cost* flow \( f \)
  - computes *fixed* tolls so that the flow \( f \) forms an approximate *equilibrium*
Largeness Assumption

For every edge $e$, the convex latency function $\ell_e$ is

- bounded by $n$ ($\ell_e(n) \leq n$)
- $\gamma$-Lipschitz

Think of $\gamma = O(1)$

Each player only has bounded influence on the latency
**Key: Algorithmic Stability**

- High-level idea: design a mediator that is “insensitive” to the reported source-destination pair of each player $i$’s deviation does not substantially
  - change the tolls
  - change the paths suggested to other players
- Formulated as *Joint Differential Privacy (JDP)* [KPRU’14]

A (randomized) mechanism $\mathcal{M} : (V \times V)^n \rightarrow \mathbb{R}_+^m \times (2^E)^n$ satisfies $\varepsilon$-joint differential privacy if for every $i$, $t_i, t'_i \in V \times V$, $t_{-i} \in (V \times V)^{n-1}$ and $O \subseteq \mathbb{R}_+^m \times (2^E)^{n-1}$

$$\Pr[\mathcal{M}(t_i, t_{-i}) \in O] \leq \exp(\varepsilon) \cdot \Pr[\mathcal{M}(t'_i, t_{-i}) \in O]$$
Misreport!

Output distribution for other players

\[ \text{ratio bounded} \]

\[ \text{Pr} [r] \]
Suppose that a mediator

- is $\epsilon$-jointly differentially private
- computes fixed tolls and path suggestions such that the resulting flow forms $\eta$-approximate equilibrium,

then good behavior forms a $O(\eta + n\epsilon)$-Nash equilibrium in the mediated game.

- Extension of [KPRU’14] and [RogersRoth’14]
Private Algorithm Layout

Input: source-destination pairs

Private Gradient Descent Method
(applicable for solving other convex programs)

Compute approximately optimal flow $f^*$

Compute fixed tolls

Private Best-Response (PBR)
let players deviate if not playing $\alpha$-best response

Some players are not satisfied

$\Psi \hat{f}$ not minimized

Output: final flow and fixed tolls

1. perturb the aggregate flow $f^*$
2. plug in the marginal-cost toll function

$\hat{\tau} = \tau^*(\hat{f})$
Main Result

For large routing game with $n$ players, there exists a mediator such that good behavior is an $O(n^{4/5})$-approximate Nash equilibrium, and the resulting flow has social cost at most

$$\text{OPT} - O(n^{4/5})$$

- For fixed network and latency of each edge growing faster than $n^{4/5}$
- Social cost is $(1 - o(1)) \cdot \text{OPT}$
- Each player playing $(1 - o(1))$-best response (multiplicative)
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