LGIC 010 & PHIL 005
Problem Set 8
Spring Term, 2019
DUE IN CLASS MONDAY, APRIL 29

- $\mathbb{Z}$ is the set of integers $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- $\mathbb{Z}^+$ is the set of positive integers $\{1, 2, 3, \ldots\}$.
- A positive integer i is a *prime number* if and only if $i > 1$ and the only divisors of i are i and 1. The first few prime numbers are $2, 3, 5, 7, 11, \ldots$.
- Let A be a structure interpreting a triadic predicate letter P. h is an *automorphism* of A if and only if h is a bijection of U^A onto U^A and for all $i, j, k \in U^A$, $\langle i, j, k\rangle \in P^A$ if and only $\langle h(i), h(j), h(k)\rangle \in P^A$.
- Let A be a structure. $\mathsf{Aut}(A) = \{h \mid h \text{ is an automorphism of } A\}$.
- Let $S(x)$ be a schema with a single free variable x and A a structure.

$$S[A] = \{a \in U^A \mid A \models S[x|a]\}.$$

1. Let A_1 be a structure interpreting a single triadic predicate letter P_1 with $U^A = \mathbb{Z}^+$ and $P_1^A = \{\langle i, j, k\rangle \mid i \cdot j = k\}$, the product relation on $\mathbb{Z}^+$.
   (a) (10 points) Let $X_1 = \{i \in \mathbb{Z}^+ \mid i \text{ is a positive power of } 2\} = \{2, 4, 8, 16, \ldots\}$. Is X_1 definable in A_1? If it is, write down a schema $S_1(x)$ such that $S_1[A_1] = X_1$; if not, specify an $h_1 \in \mathsf{Aut}(A_1)$ such that $h_1[X_1] \neq X_1$.
   (b) (10 points) Let $X_2 = \{i \in \mathbb{Z}^+ \mid i \text{ is a positive power of a prime number}\} = \{2, 4, \ldots, 3, 9, \ldots\}$. Is X_2 definable in A_1? If it is, write down a schema $S_2(x)$ such that $S_2[A_1] = X_2$; if not, specify an $h_2 \in \mathsf{Aut}(A_1)$ such that $h_2[X_2] \neq X_2$.

2. Let A_2 be a structure interpreting a single triadic predicate letter P_2 with $U^A = \mathbb{Z}$ and $P_2^A = \{\langle i, j, k\rangle \mid i + j = k\}$, the sum relation on $\mathbb{Z}$.
   (a) (10 points) Let $X_3 = \{i \in \mathbb{Z} \mid 0 < i\}$. Is X_3 definable in A_2? If it is, write down a schema $S_3(x)$ such that $S_3[A_2] = X_3$; if not, specify an $h_3 \in \mathsf{Aut}(A_2)$ such that $h_3[X_3] \neq X_3$.
   (b) (10 points) Let $X_4 = \{i \in \mathbb{Z} \mid i \text{ is divisible by 5 (without remainder)}\}$. Is X_4 definable in A_2? If it is, write down a schema $S_4(x)$ such that $S_4[A_2] = X_4$; if not, specify an $h_4 \in \mathsf{Aut}(A_2)$ such that $h_4[X_4] \neq X_4$.

For each of the following pairs consisting of a set of schemata X and a schema S determine whether X implies S. If so, provide a cogent argument to establish the implication. If not, specify a structure which makes S false and all the schemata in X true. Each problem is worth 25 points.

3. (20 points) $X : \{(\forall x)(\forall y)(\exists z)(\forall w)(Rxyw \equiv z = w),$
$(\forall x)(\forall y)(\forall z)(\forall w)(\forall v)((Rxyv \wedge Rzwv) \supset (x = z \wedge y = w))\}$
$S : (\exists x)(\forall y)x = y$

4. (20 points) $X : \{x_i \neq x_j \mid i < j \text{ and } i, j \in \mathbb{Z}^+\} \cup \{(\forall x)\neg Lxx,$
$(\forall x)(\forall y)(Lxy \supset (Lyz \supset Lxz)), (\forall x)(\forall y)(x \neq y \supset (Lxy \vee Lyx),$
$(\forall x)((\exists y)Lyx \supset (\exists y)(Lyx \wedge (\forall z)\neg(Lyz \wedge Lzx))),$
$(\forall x)((\exists y)Lxy \supset (\exists y)(Lxy \wedge (\forall z)\neg(Lzy \wedge Lxz))),$
$(\exists x)(\forall y)\neg Lyx, (\exists x)(\forall y)\neg Lxy\}$
$S : (\forall x)x \neq x$

5. (20 points) $X : \{(\forall x)\neg Lxx, (\forall x)(\forall y)(Lxy \supset (Lyz \supset Lxz)),$
$(\forall x)(\forall y)(x \neq y \supset (Lxy \vee Lyx), (\forall x)(\exists y)(Lxy \wedge (\forall z)\neg(Lzy \wedge Lxz))),$
$(\exists x)(\forall y)\neg Lyx\}\}$
$S : (\forall x)((\exists y)Lyx \supset (\exists y)(Lyx \wedge (\forall z)\neg(Lyz \wedge Lzx)))$