LGIC 010 & PHIL 005
Problem Set 5
Spring Term, 2017
DUE IN CLASS MONDAY, MARCH 13

We deploy the following concepts in formulating some of the problems below.

- If X is a finite set, we write $|X|$ for the number of members of X.
- A *graph* is a structure interpreting one dyadic predicate letter L. A *colored graph* is a structure interpreting one dyadic predicate letter L and one monadic predicate letter F.
- We use
  - Irr to abbreviate the schema $(\forall x)\neg Lxx$,
  - Sym to abbreviate the schema $(\forall x)(\forall y)(Lxy \supset Lyx)$, and
  - SG to abbreviate the conjunction of Irr and Sym. Structures that satisfy SG are called *(colored) simple graphs.*
- If A and B are colored simple graphs, A is a *subgraph* of B if and only if $U^A \subseteq U^B$, and $L^A \subseteq L^B$ and $F^A = F^B \cap U^A$.
- The *size* of a simple graph A (written $\mathsf{size}(A)$) is $|L^A|/2$. This corresponds to the number of "undirected edges" of A.
- If S is a schema, we write $\mathsf{mod}(S, n)$ for the set of structures A such that $A \models S$ and $U^A = \{1, \ldots, n\}$.
- Let K be a set of simple graphs. We call A a *size maximal* member of K if and only if $A \in K$ and for every $B \in K$, $\mathsf{size}(A) \geq \mathsf{size}(B)$.
- For $n \geq 2$, we let $\Delta_n(x_1, \ldots, x_n)$ abbreviate the schema:

$$x_1 \neq x_2 \wedge x_1 \neq x_3 \ldots \wedge x_{n-1} \neq x_n.$$

- For $n \geq 3$, we let C_n abbreviate the schema:

$$(\exists x_1)\ldots(\exists x_n)(\Delta_n(x_1, \ldots, x_n) \wedge Lx_1x_2 \wedge Lx_2x_3 \wedge \ldots L_{n-1}x_n \wedge Lx_nx_1).$$

- A graph A is *acyclic* if and only if for every $n \geq 3$, $A \models \neg C_n$.
- A simple graph is *1-regular* if and only if it satisfies the following schema.

$$(\forall x)(\exists y)(Lxy \wedge (\forall z)(Lxz \supset z = y))$$

Instructions and a Practice Problem with Solution

In problems 1-4, you **MUST** specify each structure as indicated, that is, explicitly state the extension of a dyadic predicate as a set of ordered pairs of members of the universe of discourse and specify the extension of a monadic predicate as a subset of the universe of discourse, as is done in the solution to Problem 0. Alternative presentations (arrow diagrams, *etc.*) will **NOT** be accepted.

0. Let S_0 be the conjunction of SG and the schema

$$(\exists x)(\forall y)(Fy \equiv y = x).$$

(a) Specify a structure A_0 which is an acyclic size maximal member of $\mathsf{mod}(S_0, 3)$.

$U^{A_0} = \{1, 2, 3\}$

$L^{A_0} = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 2\rangle, \langle 2, 3\rangle\}$

$F^{A_0} = \{2\}$

(b) How many structures are acyclic size maximal members of $\mathsf{mod}(S_0, 3)$?
9

1. Let S_1 be the conjunction of the following schemata.

   - $(\forall x)(\forall y)(Lxy \supset \neg Lyx) \wedge (\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz))$
   - $(\forall x)(\forall y)(\forall z)((Lyx \wedge Lzx \wedge y \neq z) \supset (Lyz \vee Lzy)) \wedge (\exists x)(\forall y)(Lxy \vee x = y)$
   - $(\exists x)(\exists y)(\exists z)(Lxy \wedge Lyz) \wedge \neg(\exists x)(\exists y)(\exists z)(\exists w)(Lxy \wedge Lyz \wedge Lzw)$

   (a) (10 points) Specify a structure A_1 which is a member of $\mathsf{mod}(S_1, 4)$.

   $U^{A_1} =$

   $L^{A_1} =$

   (b) (10 points) What is the value of $|\mathsf{mod}(S_1, 4)|$? How many members of $\mathsf{mod}(S_1, 4)$ are acyclic?

2. Let S_2 be the conjunction of SG and the schema

$$(\forall x)(\forall y)(Lxy \supset (Fx \oplus Fy)).$$

   (a) (10 points) Specify a structure A_2 which is a size maximal member of $\mathsf{mod}(S_2, 6)$.

   $U^{A_2} =$

   $L^{A_2} =$

   $F^{A_2} =$

   (b) (10 points) How many structures are size maximal members of $\mathsf{mod}(S_2, 6)$?

   (c) (10 points) Let B be a size maximal member of $\mathsf{mod}(S_2, 6)$. How many 1-regular subgraphs C of B satisfy the condition $U^C = U^B$?

3. Let S_3 be the conjunction of the following four schemata.

- $(\forall x)(\forall y)(Lxy \supset \neg Lyx) \wedge (\forall x)(\forall y)(x \neq y \supset (Lxy \vee Lyx))$
- $(\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz))$
- $(\forall x)((\forall y)\neg Lyx \supset F(x)) \wedge (\forall x)((\forall y)\neg Lxy \supset \neg F(x))$
- $(\forall x)(Fx \equiv (\forall y)((Lxy \wedge \neg(\exists z)(Lxz \wedge Lzy)) \supset \neg Fy))$

(a) (10 points) Specify a structure $A_3 \in \mathsf{mod}(S_3, 6)$.

$U^{A_3} =$

$L^{A_3} =$

$F^{A_3} =$

(b) (10 points) What is the value of $|\mathsf{mod}(S_3, 6)|$?

4. Let S_4 be the conjunction of the following four schemata.

- $(\forall v)(\forall w)(\forall x)(\forall y)(\forall z)((Rvwz \wedge Rxyz) \supset (v = x \wedge w = y))$
- $(\forall x)(\forall y)(\forall z)(Rxyz \supset (Fx \wedge Gy))$
- $(\forall x)(\forall y)((Fx \wedge Gy) \supset (\exists z)(\forall w)(Rxyw \equiv w = z))$
- $(\forall z)(\exists x)(\exists y)Rxyz$
- $(\exists x)(\exists y)(Fx \wedge Fy \wedge x \neq y) \wedge (\exists x)(\exists y)(Gx \wedge Gy \wedge x \neq y)$

(a) (10 points) For which values of n strictly between 1 and 7 is $\mathsf{mod}(S_4, n)$ nonempty?

(b) (10 points) For the largest such value of n, specify a structure $A_4 \in \mathsf{mod}(S_4, n)$.

$U^{A_4} =$

$F^{A_4} =$

$G^{A_4} =$

$R^{A_4} =$

(c) (10 points) For the largest such value of n, what is the value of $|\mathsf{mod}(S_4, n)|$?