Preview of Lecture 04.06

On 04.06, we will prove

Theorem 1 *If $V \subseteq U^B$ is definable over B, then V is finite or V is co-finite.*

Proof: Suppose to the contrary, that there is a set V, definable over B, which is neither finite nor co-finite, and suppose that the schema $S(x)$ defines V over B. We derive a contradiction from this hypothesis. Let $\Lambda = \{S \mid B \models S\}$; Λ is the set of all schemata true in the structure B and is often called the *complete theory* of B. Let y and z be fresh variables which occur nowhere in Λ, $S(x)$, or any of the schemata $S^n(x)$ for $n \geq 0$ defined above. Define the set of schemata Γ as follows.

$$\Gamma = \Lambda \cup \{y \neq z, S(y), \neg S(z)\} \cup \{\neg S^n(y), \neg S^n(z) \mid n \geq 0\}.$$

Let Δ be a finite subset of Γ. It follows from the fact that both $S[B]$ and $\neg S[B]$ are infinite, that Δ is satisfied by B with suitable assignments from U^B to the variables y and z. Hence, by the Compactness Theorem, Γ itself is satisfiable. Of course, if the structure C satisfies Γ, then C is not isomorphic to B since the the elements of U^C assigned to y and z in C (call them a and b respectively) are not reachable in C from the unique element of C with no predecessor. We will show that there is an automorphism h of C with $h(a) = b$. This will yield the desired contradiction, since $C \models S(y|a)$ and $C \models \neg S(z|b)$. Note that B, and hence C, satisfy the following schemata.

- $(\exists x)(\forall y)((\forall z)\neg Lzy \equiv x = y)$
- $(\forall x)(\exists y)(\forall z)(Lxz \equiv z = y)$
- $(\forall x)(\forall y)(\forall z)((Lxz \wedge Lyz) \supset x = y)$
- $(\forall x)\neg Lxx$

  $\vdots$

  $(\forall x)(\forall y_1) \ldots (\forall y_n)\neg(Lxy_1 \wedge Ly_1y_2 \ldots \wedge Ly_nx)$

  $\vdots$

The first three schemata guarantee that L^C is an injective functional relation which is "almost" surjective – there is a unique element of U^C which lacks a pre-image under the function whose graph is L^C. Note that this guarantees that U^C is infinite. The final infinite list of schemata guarantee that the the function whose graph is L^C contains no finite cycles. Since C is not isomorphic to B all this implies that C consists of an L^C chain that is isomorphic to B and a non-empty set of L^C chains each of which is isomorphic to $\mathbb{Z}$ (the set of all integers) equipped with its usual successor relation. But, since a and b must lie on one or two of these "$\mathbb{Z}$-chains," there is an automorphism h of C with $h(a) = b$. ∎