Preview of Lecture 03.30 and beyond

On 03.30, we will begin to look at definability in infinite structures. We will start with an example where automorphisms can be applied to give a complete analysis of which sets are definable, and progress to examples of rigid infinite structures where automorphisms are powerless to reveal that any set is not definable. We will show that in every infinite structure, there are many sets which are not definable, thereby establishing a severe limitation on the direct application of automorphisms to analyze definability over infinite structures. This observation will lead us to a search for other techniques to analyze the definable sets of an infinite structure. This search will lead directly into the consideration of topics that will occupy our attention for the remainder of the Term. These will include the soundness and completeness of a deductive apparatus for polyadic quantification theory, and the impossibility of a decision procedure for validity of polyadic quantificational schemata. You will need to read very carefully sections 31-34, and 41 of *Deductive Logic* to prepare for classes beginning the week of April 4. These sections elaborate a deductive apparatus for polyadic quantification that is both sound and complete.

We first analyze definability in the infinite graph A described as follows:

- $U^A = \mathbb{Z}$, the set of all integers, $\{\ldots -2, -1, 0, 1, 2, \ldots\}$;
- $L^A = \{\langle i, j\rangle \mid j \text{ is the absolute value of } i\}$. (Recall that the absolute value of an integer i is i, if $i \geq 0$, and is $-i$, if $i < 0$.)

It follows that every permutation g of $\mathbb{Z}^+$ can be extended to an automorphism h of A by setting $h(i) = g(i)$, for $i \in \mathbb{Z}^+$; $h(0) = 0$; and $h(i) = -g(-i)$, for $i < 0$. Let's write $\mathbb{Z}^-$ for the set of negative integers. Thus, $\mathsf{Orbs}(A, \mathsf{Aut}(A)) = \{\mathbb{Z}^+, \{0\}, \mathbb{Z}^-\}$. Each orbit is definable:

- $S_1[A] = \mathbb{Z}^+$, where $S_1(x)$ is $(\exists y)(y \neq x \wedge Lyx)$;
- $S_2[A] = \mathbb{Z}^-$, where $S_2(x)$ is $(\forall y)\neg Lyx$;
- $S_3[A] = \{0\}$, where $S_3(x)$ is $\neg S_1(x) \wedge \neg S_2(x)$.

Hence, there are exactly eight sets definable in A:

1. $\emptyset$,
2. $\{0\}$,
3. $\mathbb{Z}^+$,
4. $\mathbb{Z}^-$,
5. $\mathbb{Z}^+ \cup \mathbb{Z}^-$,
6. $\mathbb{Z}^+ \cup \{0\}$,
7. $\mathbb{Z}^- \cup \{0\}$,

8. $\mathbb{Z}$.

We next look at another infinite structure B where definability behaves very differently. B is described as follows:

- $U^B = \mathbb{Z}^+ \cup \{0\}$;
- $L^B = \{\langle i, j\rangle \mid j = i + 1\}$.

We will see that every finite subset (and hence every co-finite subset) of U^B is definable in B. We will give a general argument to show that there are subsets X of U^B that are not definable in B. But we will also see that $\mathsf{Aut}(B) = \{e\}$, so there is no possibility of exhibiting an automorphism h of B with $h[X] \neq X$, that is, the "automorphism method" is powerless to establish the undefinability of any subset of U^B in B.