Preview of Lecture 03.21

On 03.21, we will explore a new topic, definability, but one of the primary tools we'll employ, automorphisms of structures, will be useful in various counting arguments of current interest.

Up to this point we have neglected schemata containing free variables. Today we will correct this oversight. Consider the schema

$$S(x): \quad (\exists y)(\forall z)(Lxz \equiv z = y).$$

Let A be a graph. We define $S[A] = \{a \in U^A \mid A \models S[x|a]\}$, that is, $S[A]$ is the set of nodes of A that satisfy the schema $S(x)$ in A when assigned to the variable x. We call $S[A]$ the *set defined by* $S(x)$ in A. In the case to hand, if A is a simple graph, then $S[A]$ is the set of nodes of A of degree 1.

Given a graph A, we will consider which subsets of U^A are *definable subsets of* A, that is for which $V \subseteq U^A$ is there a schema $S[x]$ such that $S[A] = V$. In the case of finite graphs, we will be able to give an entirely satisfactory analysis in terms of the symmetries of A, that is, the collection of automorphisms of A. Recall that h is an *automorphism* of A if and only if h is a bijection of U^A onto U^A and for all $a, b \in U^A$,

$$\langle a, b\rangle \in L^A \text{ if and only if } \langle h(a), h(b)\rangle \in L^A.$$

In other words, h is an automorphism of A if and only if h is an isomorphism of A onto itself. We define $\mathsf{Aut}(A) = \{h \mid h \text{ is an automorphism of } A\}$. The following theorem is fundamental.

Theorem 1 *Let A be a graph and $h \in \mathsf{Aut}(A)$. For every $a \in U^A$ and every schema $S(x)$,*

$$A \models S[x|a] \textit{ if and only if } A \models S[x|h(a)].$$

If f is a function with domain U and $V \subseteq U$, we define $f[V] = \{f(a) \mid a \in V\}$ (the f *image* of V). With this notation in hand, we can now state a corollary to Theorem 1 which bears on definability.

Corollary 1 *Let A be a graph and $h \in \mathsf{Aut}(A)$. If V is a definable subset of A, then $h[V] = V$.*

Thus, in order to show that V is *not* a definable subset of A is suffices to exhibit an $h \in \mathsf{Aut}(A)$ and $a \in V$ such that $h(a) \notin V$. Moreover, in the case of finite structures, the converse of Corollary 1 is true.

Theorem 2 *Let A be a finite graph and $V \subseteq U^A$. V is a definable subset of A, if for every $h \in \mathsf{Aut}(A)$, $h[V] = V$.*

In order to prove Theorem 2, and to apply it to questions of counting definable sets, it will be useful to introduce the notion of the *orbit of a node* $a \in U^A$ *under the action of* $\mathsf{Aut}(A)$:

$$\mathsf{orb}(a, A) = \{h(a) \mid h \in \mathsf{Aut}(A)\}.$$

We define $\mathsf{Orbs}(A) = \{\mathsf{orb}(a, A) \mid a \in U^A\}$. As a corollary to Corollary 1 and Theorem 2 we have:

Corollary 2 *Let A be a finite graph and $V \subseteq U^A$. V is a definable subset of A if and only if either $V = \emptyset$ or there is a sequence of sets $O_1, \ldots, O_k$, where each $O_i \in \mathsf{Orbs}(A)$, and $V = O_1 \cup \ldots \cup O_k$.*

It follows at once from Corollary 2, that if A is a finite graph, then the number of definable subsets of A is $2^{|\mathsf{Orbs}(A)|}$. We will analyze a few examples in class, using the tools we've developed.

We will also look at another application of these ideas, namely to counting structures "up to isomorphism." We refresh our memories concerning some important ideas introduced in *Memoir 13*.

- Let A and B be graphs and let f be a function with domain U^A and range U^B. f is an *isomorphism from A onto B* if and only if f is a bijection and for all $i, j \in U^A$, $\langle i, j \rangle \in L^A$ if and only if $\langle f(i), f(j) \rangle \in L^B$.

- Let A be a graph with $U^A = [n]$ and let f be a permutation of $[n]$, that is, a bijection of $[n]$ onto $[n]$. We define $f[A]$ to be the graph with universe $[n]$ such that for all $i, j \in [n]$, $\langle f(i), f(j) \rangle \in L^{f([A])}$ if and only if $\langle i, j \rangle \in L^A$.

- Let A and B be graphs. A is *isomorphic to* B if and only if there is an isomorphism f from A onto B.

Let $\mathbb{S}_n$ be the set of permutations of $[n]$ and let $\mathbb{D}_n$ be the set of graphs A with $U^A = [n]$. Observe at once that for every $A \in \mathbb{D}_n$ and $h \in \mathbb{S}_n$, h is an isomorphism from A onto $h[A]$. In analogy with our discussion of the "action" of $\mathsf{Aut}(A)$ on U^A, we may think of $\mathbb{S}_n$ *acting on* $\mathbb{D}_n$. In particular, for $A \in \mathbb{D}_n$, we define $\mathsf{orb}(A, \mathbb{S}_n) = \{h[A] \mid h \in \mathbb{S}_n\}$. The following result is a special case of the *Orbit-Stabilizer Theorem*.

Theorem 3 *For all $A \in \mathbb{D}_n$,*

$$|\mathbb{S}_n| = |\mathsf{orb}(A, \mathbb{S}_n)| \cdot |\mathsf{Aut}(A)|.$$

We will discuss various applications of Theorem 3 to calculating $|\mathsf{mod}(S, n)|$, and related counting problems.