Preview of Lecture 03.14

On 03.14, we will discuss another interesting aspect of the expressive power of polyadic quantification theory. We write $\mathbb{Z}^+$ for the set of positive integers $\{1, 2, 3, \ldots\}$. The *spectrum* of a schema S (written $\mathsf{Spec}(S)$) is defined as follows.

$$\mathsf{Spec}(S) = \{n \in \mathbb{Z}^+ \mid \mathsf{mod}(S, n) \neq \emptyset\}.$$

Recall the schema $\mathsf{SG} \wedge \mathsf{1reg}$ which defines the collection of 1-regular simple graphs. We have already noticed that $\mathsf{Spec}(\mathsf{SG} \wedge \mathsf{1reg})$ is the set of even numbers, that is, $\mathsf{Spec}(\mathsf{SG} \wedge \mathsf{1reg}) = \{2i \mid i \in \mathbb{Z}^+\}$.

Let's look at another important class of graphs, namely, equivalence relations, and see how they can be put to use in generating schemata with a wide range of spectra. A graph A is an *equivalence relation* if and only if L^A is reflexive, symmetric, and transitive, that is, if and only if $A \models \mathsf{Eq}$, where Eq is the conjunction of the following schemata.

- Refl: $(\forall x)Lxx$
- Sym: $(\forall x)(\forall y)(Lxy \supset Lyx)$
- Trans: $(\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz))$

Now suppose we'd like to construct a schema S such that

- S implies Eq, and
- $\mathsf{Spec}(S) = \{3i + 1 \mid i \in \mathbb{Z}^+ \cup \{0\}\}$.

The easiest way to meet the first condition is to formulate S as a conjunction, one conjunct of which is Eq itself. But what more should we say? Well, the universe U^A of an equivalence relation A is partitioned into mutually disjoint *equivalence classes* by the relation L^A; for each $a \in U^A$, the equivalence class $\hat{a}$ of a, is $\{b \in U^A \mid \langle a, b \rangle \in L^A\}$. Now if we can construct a schema T that says every equivalence class but one is of size three, and that the exceptional equivalence class is of size one, then we may take S to be the conjunction of Eq and T. The following schema T does the job.

$$\begin{aligned}
&(\exists x)(\forall t)((\forall y)(Lty \supset y = t) \equiv x = t) \wedge \\
&\quad (\forall z)((\exists r)(r \neq z \wedge Lrz) \supset \\
&\quad (\exists v)(\exists w)(v \neq z \wedge v \neq w \wedge w \neq z \wedge (\forall u)(Luz \equiv (u = z \vee u = v \vee u = w))))
\end{aligned}$$

We will go on to explore further examples of spectra. Note that in general it is not the case that $\mathsf{Spec}(\neg S) = \mathbb{Z}^+ - \mathsf{Spec}(S)$. Convince yourself by constructing some examples where the equation fails! Can you think of examples where the equation holds? Can you think of a general condition on $\mathsf{Spec}(S)$ that guarantees the failure of the equation? We will discuss these questions in class on Monday.