23 Lecture 04.20

We considered the problem of establishing that a schema S is not implied by a set of schemata X, or equivalently, that the set of schemata $X \cup \{\neg S\}$ is not satisfiable. As we noted last time, there is no uniform approach to this problem, that is, the collection of satisfiable schemata is *not* semi-decidable.

Let X be the conjunction of the following schemata.

- $(\forall x)(\forall y)(\forall z)((Lxy \wedge Lyz) \supset Lxz)$
- $(\forall x)(\forall y)(x \neq y \supset (Lxy \vee Lyx))$
- $(\forall x)\neg Lxx$
- $(\forall x)(\exists y)(Lxy \wedge (\forall z)\neg(Lxz \wedge Lzy))$
- $(\forall x)(\exists y)(Lyx \wedge (\forall z)\neg(Lyz \wedge Lzx))$
- $(\forall x)(\exists y)(Lyx \wedge Fy)$
- $(\forall x)(\exists y)(Lxy \wedge Fy)$
- $(\forall x)(\forall y)((Fx \wedge Fy \wedge Lxy) \supset (\exists z)(Fz \wedge Lxz \wedge Lzy))$

We showed that $X \not\models (\forall x)Lxx$, that is, we showed X is satisfiable by constructing a structure A with $A \models X$. The structure A is defined as follows. Recall that $\mathbb{Z}$ is the set of integers and $\mathbb{Q}^+$ is the set of positive rational numbers.

- $U^A = \mathbb{Q}^+ \times \mathbb{Z} = \{\langle r, i\rangle \mid r \in \mathbb{Q}^+ \text{ and } i \in \mathbb{Z}\}$ (the cartesian product of $\mathbb{Q}^+$ and $\mathbb{Z}$).
- $L^A = \{\langle\langle r, i\rangle, \langle s, j\rangle\rangle \mid r < s\} \cup \{\langle\langle r, i\rangle, \langle s, j\rangle\rangle \mid r = s \text{ and } i < j\}$.

We gave another example of demonstrating satisfiability, this time for an infinite collection of schemata. Let S be the conjunction of the following schemata.

- $(\forall x)(\forall y)(\forall z)((Lxy \wedge Lyz) \supset Lxz)$
- $(\forall x)(\forall y)(x \neq y \supset (Lxy \vee Lyx))$
- $(\forall x)\neg Lxx$
- $(\forall x)((\exists y)Lxy \supset (\exists y)(Lxy \wedge (\forall z)\neg(Lxz \wedge Lzy)))$
- $(\forall x)((\exists y)Lyx \supset (\exists y)(Lyx \wedge (\forall z)\neg(Lyz \wedge Lzx)))$
- $\neg(\forall x)(\exists y)Lyx$
- $\neg(\forall x)(\exists y)Lxy$

For each $n \geq 2$, let R^n be the schema,

$$(\exists x_1)\ldots(\exists x_n)\bigwedge_{1\leq i<j\leq n} Lx_ix_j.$$

Finally, let $X = \{S\} \cup \{R^n \mid n \geq 2\}$. We gave two proofs that X is satisfiable. The first appealed to the

Theorem 1 (Compactness Theorem) *Let Σ be a set of schemata of polyadic quantification theory. If every finite $\Delta \subseteq \Sigma$ is satisfiable, then Σ is satisfiable.*

First Proof: Observe that for every $n \geq 2$, $\{S\}\cup\{R^m \mid m \leq n\}$ is satisfied by a linear order of length n. Hence, by the Compactness Theorem, X is satisfiable. ■

Second Proof: Define the structure B as follows.

- $U^B = \mathbb{Z}$.
- $L^B = \{\langle i, j\rangle \mid (0 \leq i \text{ and } j < 0) \text{ or } (i < j \text{ and } (0 \leq i, j \text{ or } i, j < 0))\}$.

Observe that $B \models X$. ■